

# Strict deformation quantization of locally convex algebras and modules

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## Abstract

In this work various symbol spaces with values in a sequentially complete locally convex vector space are introduced and discussed. They are used to define vector-valued oscillatory integrals which allow to extend Rieffel's strict deformation quantization to the framework of sequentially complete locally convex algebras and modules with separately continuous products and module structures, making use of polynomially bounded actions of  $\mathbb{R}^n$ . Several well-known integral formulas for star products are shown to fit into this general setting, and a new class of examples involving compactly supported  $\mathbb{R}^n$ -actions on  $\mathbb{R}^n$  is constructed.

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# 1 Introduction

Deformation quantization as introduced in [2] comes in several different flavours: in formal deformation quantization one deforms the commutative pointwise product of the Poisson algebra of smooth functions on a Poisson manifold into a noncommutative *star product* as a formal associative deformation in the sense of Gerstenhaber [14] with deformation parameter  $\hbar$ . Here the general existence and classification for arbitrary Poisson manifolds is known and follows from Kontsevich's formality theorem [22], see [24] for an introductory textbook.

However, for many reasons formal deformations are not sufficient: for the original application to quantum mechanics one has to treat  $\hbar$  as a positive number and not just as a formal parameter. But also applications beyond quantum theory require a more analytic framework. In particular, deformation quantization provides fundamental examples in noncommutative geometry where a  $C^*$ -algebraic formulation is needed.

In [23], Rieffel introduced a very general way to construct  $C^*$ -algebraic deformations based on a strongly continuous action of  $\mathbb{R}^d$  on a  $C^*$ -algebra  $\mathcal{A}$ . For the smooth vectors  $\mathcal{A}^\infty$  with respect to the action a product formula based on an oscillatory integral was established, generalizing the well-known Weyl quantization of  $\mathbb{R}^{2n}$ . In a second step, a matching  $C^*$ -norm is constructed leading to a continuous field of  $C^*$ -algebras over the parameter space of  $\hbar \in \mathbb{R}$ . This construction and variants of it have by now found many applications in noncommutative geometry [10, 13, 23] and quantum physics, in particular in the context of quantum field theory on noncommutative spacetimes [1, 8, 11, 17, 19].

While for the construction of deformed  $C^*$ -algebras Rieffel's work is sufficient, it turns out that the first step of deforming the smooth vectors  $\mathcal{A}^\infty$  is of interest already for its own sake: Rieffel worked with a Fréchet algebra with an isometric action.

It is this situation which we want to generalize in various directions in the present work. First the restriction to a Fréchet algebra has to be overcome as there are several examples of interest which do not fall into this class. When interested in noncommutative spacetimes, a smooth structure in form of a deformation of the smooth functions is needed for many reasons. Thus we are interested in e.g. deformations of  $\mathcal{C}_0^\infty(M)$ . Moreover, together with a deformation of algebras one is interested in a possible deformation of modules as well. In the above example one might also be interested in a corresponding deformation of the distribution spaces  $\mathcal{C}_0^\infty(M)'$ . Hence we clearly have to pass beyond a Fréchet situation. Here we face several new phenomena: first of all products or module structures may be separately continuous without being continuous. In fact, there are many natural examples like this. Second, sequentially complete locally convex spaces need not be complete, with the distribution spaces as the most prominent examples. Third, the restriction to isometric actions, which is natural in the original  $C^*$ -setting, seems to be too restrictive in a more general locally convex framework. Again, many examples of interest show that one has to overcome this restriction.

As is well known, scalar-valued oscillatory integrals can be defined for more general functions than smooth functions with bounded derivatives – here the Hörmander symbols are a natural candidate. Thus we will adapt the notion of a symbol to the vector-valued case and study oscillatory integrals. These will be needed to handle actions of  $\mathbb{R}^n$  which are not isometric but satisfy certain polynomial growth conditions.

Compared to the scalar case the new feature is that for every continuous seminorm (of a defining system of seminorms) of the target space we have to allow for a specific growth. The examples show that we cannot expect to have a uniform growth for all seminorms.

The main result of this work is the construction of a Rieffel deformation for a sequentially complete locally convex algebra with a separately continuous product with respect to a smooth polynomially bounded action of  $\mathbb{R}^n$  by automorphisms. Analogously, we give the corresponding deformation for a sequentially complete locally convex module with separately continuous module structure, provided the module structure is covariant for the  $\mathbb{R}^n$ -action. To this end we introduce the relevant symbol spaces and their oscillatory integrals based on a Riemann integral as we want to include sequentially complete spaces as well. This part is clearly of independent interest. We discuss several known examples within this framework and provide one new example of an action of  $\mathbb{R}^n$  with compact support. A priori one can only guarantee exponential bounds for the derivatives of such an action, but by a particular construction we achieve polynomial growth behaviour. Actions of this type are needed in models of locally noncommutative spacetimes as introduced in [1, 19]. In fact, the wish to have a smooth version of [19] was one of the main motivations to develop the above generalization of Rieffel's original work as a compactly supported action cannot be expected to be isometric for the seminorms of smooth functions. In the diploma thesis [18, Sect. 6.2] some aspects of the vector-valued oscillatory integrals were already anticipated.

It should be mentioned that there are still generalizations possible. One important step beyond Rieffel's original setting is to include actions of other Lie groups than  $\mathbb{R}^n$ . Here one first needs to find an analogue of Weyl quantization which then serves as universal deformation formula. This point of view was taken in the works of Bieliavsky et al., see e.g. [3–5, 7]. While these works mainly deal with the  $C^*$ -algebraic deformation, in a more recent work [6], Bieliavsky and Gayral discuss also deformation aspects of Fréchet algebras based on symbol spaces and oscillatory integrals similar to ours. We leave it to a future investigation of whether their construction can be extended beyond the Fréchet case: in principal this looks very promising.

The paper is organized as follows. In Section 2 we introduce vector-valued symbols in the spirit of Hörmander symbols. However, the order as well as the type of the symbol may depend on the seminorm of the target space, a generalization needed to deal with the examples discussed later. We introduce in detail various symbol spaces, investigate the continuity properties of the usual algebraic manipulations, and show that the affine symmetries of the domain give continuous group actions on the symbol spaces. In particular, the translations act smoothly. In Section 3 we discuss the oscillatory integrals. Our approach follows the usual scalar case with the technical complication that we have to deal with many seminorms on the target instead of one. Thus a careful investigation of the polynomial growth is presented. The integrals are based on a Riemann integral for the smooth compactly supported functions as we want to include targets which are only sequentially complete. After this preparation, Section 4 is devoted to the deformation program. Based on the developed oscillatory integrals we extend Rieffel's construction to actions of  $\mathbb{R}^n$  by automorphisms on sequentially complete locally convex algebras with separately continuous products and their modules. In case where the products are continuous also the resulting deformed products are continuous. Finally, Section 5

contains the examples. First we discuss the usual action of  $\mathbb{R}^{2n}$  on itself by translations and the induced action on various function spaces. Here in particular the scalar symbol spaces, the Schwartz space, and certain distribution spaces are discussed. This way we show that the well-known Weyl product formula, being defined pointwise for these spaces, can be understood as resulting from the oscillatory integral formulas. This is a nontrivial statement as in all cases the action is *not* isometric. The second example will be used in a future project for the construction of locally noncommutative spacetimes and corresponding quantum field theory models. It provides an action of  $\mathbb{R}^n$  on  $\mathbb{R}^n$  with compact support inside a given compact subset such that the induced action on the smooth functions is polynomially bounded. The difficulty is to pass from a trivially given exponential growth of the derivatives to a polynomial growth.

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## 2 Vector-valued symbols

In this section we introduce vector-valued symbols as smooth functions  $F : \mathbb{R}^n \rightarrow V$  which take values in a sequentially complete locally convex vector space  $V$  over  $\mathbb{C}$  and satisfy polynomial growth conditions for all their derivatives. At the present stage, all definitions also make sense for a  $V$  being a real vector space only, but as soon as we discuss the oscillatory integral, complex phases will enter the game. In the case of scalar functions, with target space  $V = \mathbb{C}$ , these spaces are closely related to Hörmander’s symbol classes [21, Section 7.8], see also [16, 20].

**Definition 2.1** *Let  $V$  be a sequentially complete locally convex space and let  $F \in \mathcal{C}^k(\mathbb{R}^n, V)$  where  $k \in \mathbb{N}_0 \cup \{+\infty\}$ . Then for every continuous seminorm  $q$  on  $V$ , every multiindex  $\mu \in \mathbb{N}_0^n$  with  $|\mu| \leq k$ , and  $m, \rho \in \mathbb{R}$ , we define*

$$\|F\|_{q,\mu}^{m,\rho} := \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \in [0, +\infty]. \quad (2.1)$$

The quantity  $\|F\|_{q,\mu}^{m,\rho}$  controls how the  $\mu$ -th derivative of  $F$  grows at infinity with respect to the seminorm  $q$ , compared to a polynomial of order  $m$ : A polynomial  $P \in V[x_1, \dots, x_n]$  of order  $m$  clearly satisfies  $\|P\|_{q,\mu}^{m,1} < \infty$  for all multiindices  $\mu$ , and  $\|F\|_{q,\mu}^{m,\rho} < \infty$  with  $\rho > 1$  (respectively  $\rho < 1$ ) indicates that the derivatives of  $F$  grow slower (respectively faster) than those of a polynomial.

In order to define the symbol classes we now fix a defining system  $\mathcal{Q}$  of continuous seminorms on  $V$ . The canonical choice is of course to take all continuous seminorms, but sometimes it will be advantageous to take only a small and manageable system. The

following definitions will formally depend on this choice, but the effect is only minor. Later we will see that the oscillatory integrals are independent of the particular choice of  $\mathcal{Q}$ .

We assign to every  $q \in \mathcal{Q}$  real numbers  $\mathbf{m}(q)$  and  $\boldsymbol{\rho}(q)$ . The corresponding map

$$\mathbf{m}: \mathcal{Q} \ni q \mapsto \mathbf{m}(q) \in \mathbb{R} \quad (2.2)$$

will be called an *order* for  $\mathcal{Q}$  and the map

$$\boldsymbol{\rho}: \mathcal{Q} \ni q \mapsto \boldsymbol{\rho}(q) \in \mathbb{R} \quad (2.3)$$

is referred to as a *type* for  $\mathcal{Q}$ .

The natural ordering of  $\mathbb{R}$  induces one for the set of all orders as well as for the set of all types. For two orders  $\mathbf{m}, \mathbf{m}'$  we write

$$\mathbf{m} \leq \mathbf{m}' \quad \text{if} \quad \mathbf{m}(q) \leq \mathbf{m}'(q) \quad (2.4)$$

for all  $q \in \mathcal{Q}$ . Then “ $\leq$ ” makes the set of all orders a directed set, and we also write  $\mathbf{m} < \mathbf{m}'$  if  $\mathbf{m}(q) < \mathbf{m}'(q)$  for all  $q \in \mathcal{Q}$ .

If we set  $\mathbf{m}(q) := m \in \mathbb{R}$  for all  $q \in \mathcal{Q}$  we get an order, called the *constant order*, and analogously the constant type  $\boldsymbol{\rho}(q) = \rho \in \mathbb{R}$ . It will turn out that this is usually too restrictive and we need more freedom in choosing an order and a type. More generally, an order  $\mathbf{m}$  is called *bounded* from above or from below by some number  $\alpha$  or  $\beta$  if for all  $q \in \mathcal{Q}$  one has  $\mathbf{m}(q) \leq \alpha$  or  $\mathbf{m}(q) \geq \beta$ , respectively.

In the following it will be reasonable to ask for the condition

$$\mathbf{m}(cq) = \mathbf{m}(q) \quad (2.5)$$

whenever  $q, cq \in \mathcal{Q}$  for a constant  $c > 0$ . In particular, for a Banach space  $V$  we usually take only the constant orders by specifying their value on the norm. If  $V$  is a Fréchet space, we will usually take a countable system  $\mathcal{Q}$ , and often consider unbounded orders.

**Definition 2.2 (Symbols)** *Let  $V$  be a sequentially complete locally convex space with defining system of seminorms  $\mathcal{Q}$ , and let  $\mathbf{m}$  and  $\boldsymbol{\rho}$  be an order and a type for  $\mathcal{Q}$ .*

- i.) A function  $F \in \mathcal{C}^\infty(\mathbb{R}^n, V)$  is called a symbol of order  $\mathbf{m}$  and type  $\boldsymbol{\rho}$  if for all  $q \in \mathcal{Q}$  and  $\mu \in \mathbb{N}_0^n$  one has*

$$\|F\|_{q,\mu}^{\mathbf{m},\boldsymbol{\rho}} := \|F\|_{q,\mu}^{\mathbf{m}(q),\boldsymbol{\rho}(q)} < \infty. \quad (2.6)$$

- ii.) The set of all symbols of order  $\mathbf{m}$  and type  $\boldsymbol{\rho}$  is denoted by  $S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ .*

Sometimes we will abbreviate the space of symbols simply by  $S^{\mathbf{m},\boldsymbol{\rho}}(V)$  or even by  $S^{\mathbf{m},\boldsymbol{\rho}}$  if  $V$  is clear from the context. However, note that  $S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$  still depends on the choice of  $\mathcal{Q}$ .

It is clear that the  $\|\cdot\|_{q,\mu}^{\mathbf{m},\boldsymbol{\rho}}$  are seminorms on  $S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ . We will endow the space of symbols with the corresponding locally convex topology, called the  *$S^{\mathbf{m},\boldsymbol{\rho}}$ -topology*. This makes  $S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$  a *Hausdorff* locally convex space since  $V$  is Hausdorff and the prefactor  $(1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(q) - \boldsymbol{\rho}(q)|\mu|)}$  is nowhere vanishing.

In the following, the smooth functions  $\mathcal{C}^\infty(\mathbb{R}^n, V)$  will always be equipped with the topology determined by all seminorms

$$p_{K,\ell,q}(F) := \sup_{\substack{x \in K \\ |\mu| \leq \ell}} q(\partial_x^\mu F(x)), \quad (2.7)$$

where  $K$  runs over compact subsets of  $\mathbb{R}^n$ ,  $\ell \in \mathbb{N}_0$ , and  $q \in \mathcal{Q}$ , and the smooth functions of compact support  $\mathcal{C}_0^\infty(\mathbb{R}^n, V)$  carry their usual inductive limit topology.

**Proposition 2.3** *Let  $V$  be a sequentially complete locally convex space with a defining system of seminorms  $\mathcal{Q}$ , and let  $\mathbf{m}$  and  $\boldsymbol{\rho}$  be an order and a type for  $\mathcal{Q}$ .*

*i.) We have continuous inclusions*

$$\mathcal{C}_0^\infty(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, V). \quad (2.8)$$

*ii.) The symbols  $S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$  are dense in  $\mathcal{C}^\infty(\mathbb{R}^n, V)$ .*

*iii.) The symbols  $S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$  are sequentially complete and even complete if  $V$  is complete.*

*iv.) For  $\mathbf{m} \leq \mathbf{m}'$  and  $\boldsymbol{\rho} \geq \boldsymbol{\rho}'$  we have the continuous inclusion*

$$S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, V). \quad (2.9)$$

*More precisely, we have for all  $F \in S^{\mathbf{m},\boldsymbol{\rho}}$ , all  $q \in \mathcal{Q}$ , and all  $\mu \in \mathbb{N}_0^n$*

$$\|F\|_{q,\mu}^{\mathbf{m}',\boldsymbol{\rho}'} \leq \|F\|_{q,\mu}^{\mathbf{m},\boldsymbol{\rho}}. \quad (2.10)$$

*Proof.* Clearly, we have a set-theoretic inclusion in (2.8) as compactly supported functions decay fast enough to have finite symbol norms (2.1) for any choices of the orders or types. With a compact set  $K \subseteq \mathbb{R}^n$ , and  $F \in \mathcal{C}_K^\infty(\mathbb{R}^n, V)$ , we get

$$\begin{aligned} \|F\|_{q,\mu}^{\mathbf{m},\boldsymbol{\rho}} &= \max_{x \in K} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(q) - \boldsymbol{\rho}(q)|\mu|)} q\left(\frac{\partial^{|\mu|} F}{\partial x^\mu}(x)\right) \\ &\leq \max_{x \in K} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(q) - \boldsymbol{\rho}(q)|\mu|)} \cdot p_{K,|\mu|,q}(F), \end{aligned}$$

with the seminorm (2.7). Since the maximum over the first factor is finite, we see that for every compact subset  $K$ , the inclusion

$$\mathcal{C}_K^\infty(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$$

is continuous. By the universal property of the inductive limit topology of  $\mathcal{C}_0^\infty(\mathbb{R}^n, V)$ , this is equivalent to the continuity of the first inclusion in (2.8). For the second inclusion, we fix a compact subset  $K \subseteq \mathbb{R}^n$  as well as  $\ell \in \mathbb{N}_0$  and  $q \in \mathcal{Q}$ . Then for a symbol  $F \in S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$  we have

$$p_{K,\ell,q}(F) = \max_{\substack{x \in K \\ |\mu| \leq \ell}} q\left(\frac{\partial^{|\mu|} F}{\partial x^\mu}(x)\right)$$

$$\begin{aligned}
&\leq \max_{\substack{x \in K \\ |\mu| \leq \ell}} (1 + \|x\|^2)^{\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \sup_{\substack{x \in \mathbb{R}^n \\ |\mu| \leq \ell}} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \mathbf{q} \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\
&= c \max_{|\mu| \leq \ell} \|F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}},
\end{aligned}$$

with a constant  $c > 0$ , where we used the fact that the continuous function  $x \mapsto (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)}$  is nowhere vanishing and hence has a locally bounded inverse. This shows the continuity of the second inclusion in (2.8). But then the second part is clear since already  $\mathcal{C}_0^\infty(\mathbb{R}^n, V)$  is dense in  $\mathcal{C}^\infty(\mathbb{R}^n, V)$ . In order to show sequential completeness, let  $(F_i)_{i \in \mathbb{N}}$  be a Cauchy sequence in  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ . Since the  $\mathcal{C}^\infty$ -topology is coarser than the  $S^{\mathbf{m}, \boldsymbol{\rho}}$ -topology by the first part, and since  $\mathcal{C}^\infty(\mathbb{R}^n, V)$  is sequentially complete, we get convergence  $F_i \rightarrow F$  to some smooth function  $F \in \mathcal{C}^\infty(\mathbb{R}^n, V)$  in the  $\mathcal{C}^\infty$ -topology. We have to show that  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  and  $F_i \rightarrow F$  in the  $S^{\mathbf{m}, \boldsymbol{\rho}}$ -topology. Thus let  $\epsilon > 0$ ,  $\mathbf{q} \in \mathcal{Q}$ , and  $\mu \in \mathbb{N}_0^n$  be given. Then fix  $N \in \mathbb{N}_0$  such that  $i, j \geq N$  gives  $\|F_i - F_j\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} < \epsilon$  by the Cauchy condition. For a point  $x \in \mathbb{R}^n$  we have by the pointwise convergence  $\frac{\partial^{|\mu|} F_i}{\partial x^\mu}(x) \rightarrow \frac{\partial^{|\mu|} F}{\partial x^\mu}(x)$  a  $N' \geq N$  depending on  $x$  such that

$$(1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \mathbf{q} \left( \frac{\partial^{|\mu|} F_j}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) < \epsilon \quad (*)$$

for all  $j \geq N'$ . Thus for  $i \geq N$  we get

$$\begin{aligned}
&(1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \mathbf{q} \left( \frac{\partial^{|\mu|} (F - F_i)}{\partial x^\mu}(x) \right) \\
&\leq (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \left( \mathbf{q} \left( \frac{\partial^{|\mu|} (F - F_j)}{\partial x^\mu}(x) \right) + \mathbf{q} \left( \frac{\partial^{|\mu|} (F_j - F_i)}{\partial x^\mu}(x) \right) \right) \\
&\stackrel{(*)}{\leq} \epsilon + \|F_i - F_j\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} \\
&\leq 2\epsilon.
\end{aligned}$$

Since this estimate can be done for all  $x \in \mathbb{R}^n$ , we can take the supremum over all  $x \in \mathbb{R}^n$  and get  $\|F - F_i\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} \leq 2\epsilon$ . Hence  $F - F_i \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  for  $i \geq N$  and thus also  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ . Moreover, we conclude that  $F_i \rightarrow F$  in the  $S^{\mathbf{m}, \boldsymbol{\rho}}$ -topology. Clearly, if  $V$  is even complete we can repeat the argument with nets instead of sequences. For the last part, it is sufficient to show the estimate (2.10). Since for  $\mathbf{m}(\mathbf{q}) \leq \mathbf{m}'(\mathbf{q})$  and  $\boldsymbol{\rho}(\mathbf{q}) \geq \boldsymbol{\rho}'(\mathbf{q})$  we have

$$(1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \geq (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}'(\mathbf{q}) - \boldsymbol{\rho}'(\mathbf{q})|\mu|)}$$

for every point  $x \in \mathbb{R}^n$  and every  $\mu \in \mathbb{N}_0^n$ , the estimate (2.10) follows.  $\square$

In case  $V$  is a Banach space, we choose just its norm  $\|\cdot\|$  in order to define the space of symbols. In this case, the order  $m := \mathbf{m}(\|\cdot\|)$  and the type  $\rho := \boldsymbol{\rho}(\|\cdot\|)$  are just single numbers, and we write  $\|\cdot\|_\mu^{m, \rho}$  instead of  $\|\cdot\|_{\|\cdot\|, \mu}^{m, \rho}$ . However,  $S^{m, \rho}(\mathbb{R}^n, V)$  is



no longer a Banach space but a Fréchet space since we have to take care of countably many differentiations. For a Fréchet space  $V$ , we take a countable defining system of seminorms and hence an order is determined by fixing countably many numbers  $\mathbf{m}(q_n)$ . Thus, in this situation the symbols are again a Fréchet space.

Note that the inclusion  $\mathcal{C}_0^\infty(\mathbb{R}^n, V) \subset S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$  is in general not (sequentially) dense in the  $S^{\mathbf{m}, \rho}$ -topology. However, we will show later (Proposition 2.10, *iv.*), that it is dense in a weaker topology. As a preparation for this, we need to study the multiplication properties of symbols.

## 2.1 Operations with symbols

In this subsection we discuss several operations one can perform with symbols, like differentiation, multiplication, composition with linear maps, and restriction. We begin by showing that the topologies of the symbol spaces are compatible with differentiation.

**Proposition 2.4** *Let  $V$  be a sequentially complete locally convex space with a defining system of seminorms  $\mathcal{Q}$ , and  $\mathbf{m}, \rho$  an order and a type for  $\mathcal{Q}$ . Then the partial derivatives are continuous linear maps*

$$\frac{\partial^{|\nu|}}{\partial x^\nu} : S^{\mathbf{m}, \rho}(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m} - \rho|\nu|, \rho}(\mathbb{R}^n, V). \quad (2.11)$$

More precisely, we have for all  $\mu \in \mathbb{N}_0^n$  and  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$

$$\left\| \frac{\partial^{|\nu|}}{\partial x^\nu} F \right\|_{\mathbf{q}, \mu}^{\mathbf{m} - \rho|\nu|, \rho} = \|F\|_{\mathbf{q}, \mu + \nu}^{\mathbf{m}, \rho}. \quad (2.12)$$

Proof. We only have to show the second statement which is clear from the definition.  $\square$

For a general discussion of multiplication of symbols, we now consider three sequentially complete locally convex spaces  $V$ ,  $W$ , and  $U$  together with a bilinear map

$$\mu : V \times W \longrightarrow U. \quad (2.13)$$

For simplicity, we require that  $\mu$  is continuous and not just separately continuous or sequentially continuous. In many applications, this will be the case. Now we fix a defining system  $\mathcal{R}$  of seminorms on  $U$  and filtrating defining systems of seminorms  $\mathcal{Q}$  and  $\mathcal{Q}'$  on  $V$  and  $W$ , respectively. Then by continuity of  $\mu$  we get for every  $r \in \mathcal{R}$  seminorms  $q \in \mathcal{Q}$  and  $q' \in \mathcal{Q}'$  and a constant  $c$  such that

$$r(\mu(v, w)) \leq c q(v) q'(w), \quad v \in V, w \in W. \quad (2.14)$$

For two orders  $\mathbf{m}$  and  $\mathbf{m}'$  on  $V$  and  $W$  we consider an order  $\mathbf{m}''$  on  $U$  such that for all  $r \in \mathcal{R}$  we have  $q \in \mathcal{Q}$  and  $q' \in \mathcal{Q}'$  such that (2.14) holds and

$$\mathbf{m}''(r) \geq \mathbf{m}(q) + \mathbf{m}'(q'). \quad (2.15)$$



In this case, we symbolically write  $\mathbf{m}'' \geq \mathbf{m} + \mathbf{m}'$  by some slight abuse of notation. Note that we relate here orders on *different* sets of seminorms (even on different spaces). Clearly, for given orders  $\mathbf{m}$  and  $\mathbf{m}'$  we can construct an order  $\mathbf{m}''$  with this property by fixing a choice of seminorms  $q(r)$  and  $q'(r)$  satisfying (2.14) and setting

$$\mathbf{m}''(r) = \mathbf{m}(q(r)) + \mathbf{m}''(q'(r)) \quad (2.16)$$

for all  $r \in \mathcal{R}$ . In the same spirit we write  $\rho'' \leq \min(\rho, \rho')$  for types  $\rho$  on  $V$ ,  $\rho'$  on  $W$ , and  $\rho''$  on  $U$ , again with respect to the continuous bilinear map  $\mu$ . With these conventions in mind, we can prove the following statement.

**Proposition 2.5** *Let  $V$ ,  $W$ , and  $U$  be sequentially complete locally convex spaces,  $\mathcal{R}$  a defining system of seminorms on  $U$ , and  $\mathcal{Q}, \mathcal{Q}'$  filtrating defining systems of seminorms on  $V, W$  respectively. Furthermore, let  $\mu: V \times W \rightarrow U$  be a continuous bilinear map.*

*i.) Pointwise evaluation of  $\mu$  gives a continuous bilinear map*

$$\mu: S^{\mathbf{m}, \rho}(\mathbb{R}^n, V) \times S^{\mathbf{m}', \rho'}(\mathbb{R}^n, W) \rightarrow S^{\mathbf{m}'', \rho''}(\mathbb{R}^n, U), \quad (2.17)$$

$$\mu(F, G)(x) := \mu(F(x), G(x)), \quad (2.18)$$

*whenever  $\mathbf{m}'' \geq \mathbf{m} + \mathbf{m}'$  and  $\rho'' \leq \min(\rho, \rho')$  with respect to  $\mu$ . More precisely, for  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$  and  $G \in S^{\mathbf{m}', \rho'}(\mathbb{R}^n, W)$  we get*

$$\|\mu(F, G)\|_{r, \kappa}^{\mathbf{m}'', \rho''} \leq 2^{|\mu|} c \max_{\nu \leq \kappa} \|F\|_{q, \nu}^{\mathbf{m}, \rho} \max_{\nu' \leq \kappa} \|G\|_{q', \nu'}^{\mathbf{m}', \rho'} \quad (2.19)$$

*whenever  $r, q$ , and  $q'$  satisfy the continuity property (2.14) with respect to  $\mu$ .*

*ii.) For  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$ ,  $G \in S^{\mathbf{m}', \rho'}(\mathbb{R}^{n'}, W)$ , let*

$$\underline{\mu}(F, G): \mathbb{R}^n \oplus \mathbb{R}^{n'} \rightarrow U, \quad \underline{\mu}(F, G)(x, y) := \mu(F(x), G(y)). \quad (2.20)$$

*Then we have a continuous bilinear map*

$$\underline{\mu}: S^{\mathbf{m}, \rho}(\mathbb{R}^n, V) \times S^{\mathbf{m}', \rho'}(\mathbb{R}^{n'}, W) \rightarrow S^{\mathbf{m}'', \rho''}(\mathbb{R}^n \oplus \mathbb{R}^{n'}, U) \quad (2.21)$$

*whenever  $\mathbf{m}'' \geq \max\{\mathbf{m}, \mathbf{m}', \mathbf{m} + \mathbf{m}'\}$  and  $\rho'' \leq \min\{0, \rho, \rho'\}$  with respect to  $\mu$ . Explicitly, for  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$ ,  $G \in S^{\mathbf{m}', \rho'}(\mathbb{R}^{n'}, W)$ , we get,  $\nu \in \mathbb{R}^n$ ,  $\nu' \in \mathbb{R}^{n'}$ ,*

$$\|\underline{\mu}(F, G)\|_{r, \nu \oplus \nu'}^{\mathbf{m}'', \rho''} \leq c \|F\|_{q, \nu}^{\mathbf{m}, \rho} \|G\|_{q', \nu'}^{\mathbf{m}', \rho'} \quad (2.22)$$

*whenever  $r, q, q'$  satisfy the continuity property (2.14) with respect to  $\mu$ .*

Proof. For *i.*), even though the formulation looks rather technical, this is essentially just the Leibniz rule. Let  $r \in \mathcal{R}$ , choose corresponding seminorms  $q \in \mathcal{Q}$  and  $q' \in \mathcal{Q}'$  satisfying (2.14), and  $\kappa \in \mathbb{N}_0^n$ . Then we have

$$\begin{aligned} & \|\mu(F, G)\|_{r, \kappa}^{\mathbf{m}'', \rho''} \\ &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}''(r) - \rho''(r)|\kappa|)} r \left( \frac{\partial^{|\kappa|} \mu(F, G)}{\partial x^\kappa}(x) \right) \end{aligned}$$

$$\begin{aligned}
&= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}''(\mathbf{r}) - \boldsymbol{\rho}''(\mathbf{r})|\kappa|)} \mathbf{r} \left( \sum_{\nu + \nu' = \kappa} \binom{\kappa}{\nu} \mu \left( \frac{\partial^{|\nu|} F}{\partial x^\nu}(x), \frac{\partial^{|\nu'|} G}{\partial x^{\nu'}}(x) \right) \right) \\
&\leq \sum_{\nu + \nu' = \kappa} \binom{\kappa}{\nu} c \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\nu|)} \mathbf{q} \left( \frac{\partial^{|\nu|} F}{\partial x^\nu}(x) \right) \\
&\quad \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}'(\mathbf{q}') - \boldsymbol{\rho}'(\mathbf{q}')|\nu'|)} \mathbf{q}' \left( \frac{\partial^{|\nu'|} G}{\partial x^{\nu'}}(x) \right) \\
&\leq 2^{|\kappa|} c \max_{\nu \leq \kappa} \|F\|_{\mathbf{q}, \nu}^{\mathbf{m}, \boldsymbol{\rho}} \max_{\nu' \leq \kappa} \|G\|_{\mathbf{q}', \nu'}^{\mathbf{m}', \boldsymbol{\rho}'},
\end{aligned}$$

since  $|\kappa| = |\nu| + |\nu'|$ . This shows (2.19), which implies the continuity of the map (2.17). For *ii.*, it is clear that for any  $F \in \mathbf{S}^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ ,  $G \in \mathbf{S}^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^{n'}, W)$ , we have  $\underline{\mu}(F, G) \in \mathcal{C}^\infty(\mathbb{R}^n \oplus \mathbb{R}^{n'}, U)$ , and that  $\underline{\mu}$  is bilinear. To prove the continuity of this map, we have to verify the bound (2.22). The estimate necessary for this is based on the fact that given  $k, k' \in \mathbb{R}$ , there holds for all  $a, b \in \mathbb{R}$

$$(1 + a^2 + b^2)^{-K} \leq (1 + a^2)^{-k} (1 + b^2)^{-k'} \quad \text{if } K \geq \max\{k, k', k + k'\}. \quad (2.23)$$

In the situation at hand, we pick seminorms  $\mathbf{r}, \mathbf{q}, \mathbf{q}'$  satisfying (2.14), multiindices  $\nu \in \mathbb{R}^n$ ,  $\nu' \in \mathbb{R}^{n'}$ , and set  $K := \frac{1}{2}(\mathbf{m}''(\mathbf{r}) - \boldsymbol{\rho}''(\mathbf{r})|\nu \oplus \nu'|)$ ,  $k := \frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\nu|)$ ,  $k' := \frac{1}{2}(\mathbf{m}'(\mathbf{q}') - \boldsymbol{\rho}'(\mathbf{q}')|\nu'|)$ . The assumptions on  $\mathbf{m}''$ ,  $\boldsymbol{\rho}''$  then guarantee that  $K \geq \max\{k, k', k + k'\}$  holds, and we can use (2.23) to estimate

$$\begin{aligned}
\|\underline{\mu}(F, G)\|_{\mathbf{r}, \nu \oplus \nu'}^{\mathbf{m}'', \boldsymbol{\rho}''} &= \sup_{\substack{x \in \mathbb{R}^n \\ y \in \mathbb{R}^{n'}}} \frac{\mathbf{r}(\partial_x^\nu \partial_y^{\nu'} \underline{\mu}(F(x), G(y)))}{(1 + \|x\|^2 + \|y\|^2)^{\frac{1}{2}(\mathbf{m}(\mathbf{r}) - \boldsymbol{\rho}(\mathbf{r})|\nu \oplus \nu'|)}} \\
&\leq \sup_{\substack{x \in \mathbb{R}^n \\ y \in \mathbb{R}^{n'}}} \frac{c \mathbf{q}(\partial_x^\nu F(x)) \mathbf{q}'(\partial_y^{\nu'} G(y))}{(1 + \|x\|^2)^{\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\nu|)} (1 + \|y\|^2)^{\frac{1}{2}(\mathbf{m}'(\mathbf{q}') - \boldsymbol{\rho}'(\mathbf{q}')|\nu'|)}} \\
&\leq c \|F\|_{\mathbf{q}, \nu}^{\mathbf{m}, \boldsymbol{\rho}} \|G\|_{\mathbf{q}', \nu'}^{\mathbf{m}', \boldsymbol{\rho}'}.
\end{aligned}$$

□

For continuous *linear* maps, a similar result holds.

**Proposition 2.6** *Let  $A: V \longrightarrow W$  be a continuous linear map between sequentially complete locally convex spaces  $V, W$  with defining systems of seminorms  $\mathcal{Q}, \mathcal{Q}'$ , and let  $\mathcal{Q}$  be filtrating. Furthermore, let orders  $\mathbf{m}$  and  $\mathbf{m}'$  and types  $\boldsymbol{\rho}$  and  $\boldsymbol{\rho}'$  for  $\mathcal{Q}$  and  $\mathcal{Q}'$  be given. Suppose for every seminorm  $\mathbf{q}' \in \mathcal{Q}'$  we find a seminorm  $\mathbf{q} \in \mathcal{Q}$  such that*

$$\mathbf{q}'(Av) \leq c \mathbf{q}(v), \quad \mathbf{m}(\mathbf{q}) \leq \mathbf{m}'(\mathbf{q}'), \quad \text{and} \quad \boldsymbol{\rho}(\mathbf{q}) \geq \boldsymbol{\rho}'(\mathbf{q}') \quad (2.24)$$

*for some  $c > 0$  and all  $v \in V$ . Then pointwise evaluation of  $A$  gives a continuous linear map*

$$A: \mathbf{S}^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathbf{S}^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^n, W). \quad (2.25)$$

More precisely, for every  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  and every  $\mu \in \mathbb{N}_0^n$ , we have

$$\|AF\|_{\mathbf{q}', \mu}^{\mathbf{m}', \boldsymbol{\rho}'} \leq c \|F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}. \quad (2.26)$$

Proof. Note that the first condition  $\mathbf{q}'(Av) \leq c \mathbf{q}(v)$  can always be satisfied since  $\mathcal{Q}$  was assumed to be filtrating and  $A$  is continuous. Thus assume that the other two requirements in (2.24) are fulfilled as well. Then we have for  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$

$$\begin{aligned} \|AF\|_{\mathbf{q}', \mu}^{\mathbf{m}', \boldsymbol{\rho}'} &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}'(\mathbf{q}') - \boldsymbol{\rho}'(\mathbf{q}')|\mu|)} \mathbf{q}' \left( \frac{\partial^{|\mu|} AF}{\partial x^\mu}(x) \right) \\ &\leq \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \mathbf{q}' \left( A \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\ &\leq c \|F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}. \end{aligned}$$

Since the seminorms  $\|\cdot\|_{\mathbf{q}', \mu}^{\mathbf{m}', \boldsymbol{\rho}'}$  determine the  $S^{\mathbf{m}', \boldsymbol{\rho}'}$ -topology, this yields the continuity of (2.25).  $\square$

The main application of Proposition 2.5, *i.*), is to multiply vector-valued symbols with scalar symbols: Choosing one target space just to be  $\mathbb{C}$ , with seminorms just the absolute value, we get for every order  $m \in \mathbb{R}$  and every type  $\rho \in \mathbb{R}$  the space of scalar symbols

$$S^{m, \rho}(\mathbb{R}^n) = S^{m, \rho}(\mathbb{R}^n, \mathbb{C}). \quad (2.27)$$

Note that here the order and the type are indeed just single numbers. We now formulate two corollaries about multiplications of symbols. Their proofs follow immediately from Proposition 2.5 and are therefore omitted.

**Corollary 2.7** *Let  $V$  be a sequentially complete locally convex space with a defining system of seminorms  $\mathcal{Q}$ .*

*i.) For all orders  $\mathbf{m}$  and types  $\boldsymbol{\rho}$  for  $\mathcal{Q}$ , and all  $m, \rho \in \mathbb{R}$ , the pointwise multiplication gives a continuous bilinear map*

$$S^{m, \rho}(\mathbb{R}^n, \mathbb{C}) \times S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m}+m, \min(\boldsymbol{\rho}, \rho)}(\mathbb{R}^n, V). \quad (2.28)$$

*ii.) In particular, if the type  $\boldsymbol{\rho}$  is bounded by  $\rho \in \mathbb{R}$ , then*

$$S^{m, \rho}(\mathbb{R}^n, \mathbb{C}) \times S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m}+m, \boldsymbol{\rho}}(\mathbb{R}^n, V) \quad (2.29)$$

*is a continuous bilinear map.*

*iii.) If  $m \leq 0$  then  $S^{m, \rho}(\mathbb{R}^n, \mathbb{C})$  is a Fréchet algebra and  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  is a continuous module over it for all bounded  $\boldsymbol{\rho} \leq \rho$ .*

**Corollary 2.8** *Let  $\mathcal{A}$  be a sequentially complete locally convex algebra with a defining system of seminorms  $\mathcal{Q}$ . Then the multiplication induces a continuous product*

$$S^{m, \rho}(\mathbb{R}^n, \mathcal{A}) \times S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^n, \mathcal{A}) \longrightarrow S^{\mathbf{m}+\mathbf{m}', \boldsymbol{\rho}}(\mathbb{R}^n, \mathcal{A}). \quad (2.30)$$

*In particular, for  $\mathbf{m} \leq 0$  the symbols  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, \mathcal{A})$  form a sequentially complete locally convex algebra themselves and any  $S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^n, \mathcal{A})$  is a sequentially complete locally convex continuous module over them.*

For later applications, we also note the following simple lemma about powers of scalar symbols.

**Lemma 2.9** *Let  $f \in S^{m,\rho}(\mathbb{R}^n, \mathbb{C})$  be a scalar symbol of order  $m$  and type  $\rho$  with  $f(x) \in \mathbb{C} \setminus [0, -\infty)$  for all  $x \in \mathbb{R}^n$ . Then for all  $\alpha \in \mathbb{C}$  with  $\operatorname{Re}(\alpha) \geq 0$  we have  $f^\alpha \in S^{\operatorname{Re}(\alpha)m, \rho}(\mathbb{R}^n, \mathbb{C})$ .*

Proof. Since  $f$  takes values only in the complex plane without the closed negative real half axis, we can use the (smooth) principal branch of the logarithm to define the powers  $f^\alpha = \exp(\alpha \log(f)) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$  for all  $\alpha \in \mathbb{C}$ . Note that we do not have to take care of the values  $\alpha \in \mathbb{N}_0$  as these particular cases are already settled by Corollary 2.8 by induction. By the chain rule we get

$$\frac{\partial^{|\mu|} f^\alpha}{\partial x^\mu} = \sum_{\substack{1 \leq k \leq |\mu| \\ \nu_1, \dots, \nu_k \in \mathbb{N}_0^n \\ |\nu_1| + \dots + |\nu_k| = |\mu|}} C_{\nu_1, \dots, \nu_k}^{k, \alpha} f^{\alpha-k} \frac{\partial^{|\nu_1|} f}{\partial x^{\nu_1}} \dots \frac{\partial^{|\nu_k|} f}{\partial x^{\nu_k}}$$

with some coefficients  $C_\nu^{k, \alpha} \in \mathbb{C}$ . We first note that for  $f^\alpha$  without derivatives we get the estimate

$$\begin{aligned} \|f^\alpha\|_0^{\operatorname{Re}(\alpha)m, \rho} &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}\operatorname{Re}(\alpha)m} |f^\alpha(x)| \\ &\leq e^{\pi|\operatorname{Im}(\alpha)|} \sup_{x \in \mathbb{R}^n} \left( (1 + \|x\|^2)^{-\frac{m}{2}} |f(x)| \right)^{\operatorname{Re}(\alpha)} \\ &= e^{\pi|\operatorname{Im}(\alpha)|} (\|f\|_0^{m, \rho})^{\operatorname{Re}(\alpha)} \end{aligned}$$

since for any complex number  $z \in \mathbb{C} \setminus [0, -\infty)$  we have  $|z^\alpha| \leq |z|^{\operatorname{Re}(\alpha)} e^{\pi|\operatorname{Im}(\alpha)|}$ , and since  $\operatorname{Re}(\alpha) \geq 0$ . Thus we need the prefactor  $(1 + \|x\|^2)^{-\frac{1}{2}\operatorname{Re}(\alpha)m}$  to compensate the growth of  $f^\alpha$ . To estimate the derivatives, we get

$$\begin{aligned} &\|f^\alpha\|_\mu^{\operatorname{Re}(\alpha)m, \rho} \\ &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\operatorname{Re}(\alpha)m - \rho|\mu|)} \left| \frac{\partial^{|\mu|} f^\alpha}{\partial x^\mu}(x) \right| \\ &\leq \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\operatorname{Re}(\alpha)m - \rho|\mu|)} \sum_{\substack{1 \leq k \leq |\mu| \\ \nu_1, \dots, \nu_k \in \mathbb{N}_0^n \\ |\nu_1| + \dots + |\nu_k| = |\mu|}} |C_{\nu_1, \dots, \nu_k}^{k, \alpha}| |f^{\alpha-k}(x)| \left| \frac{\partial^{|\nu_1|} f}{\partial x^{\nu_1}} \right| \dots \left| \frac{\partial^{|\nu_k|} f}{\partial x^{\nu_k}} \right| \\ &\leq \sum_{\substack{1 \leq k \leq |\mu| \\ \nu_1, \dots, \nu_k \in \mathbb{N}_0^n \\ |\nu_1| + \dots + |\nu_k| = |\mu|}} |C_{\nu_1, \dots, \nu_k}^{k, \alpha}| \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\operatorname{Re}(\alpha)m - km)} |f^{\alpha-k}(x)| \\ &\quad \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(m - \rho|\nu_1|)} \left| \frac{\partial^{|\nu_1|} f}{\partial x^{\nu_1}} \right| \dots \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(m - \rho|\nu_k|)} \left| \frac{\partial^{|\nu_k|} f}{\partial x^{\nu_k}} \right| \end{aligned}$$

$$\leq \sum_{\substack{1 \leq k \leq |\mu| \\ \nu_1, \dots, \nu_k \in \mathbb{N}_0^n \\ |\nu_1| + \dots + |\nu_k| = |\mu|}} |C_{\nu_1, \dots, \nu_k}^{k, \alpha}| e^{\pi |\operatorname{Im}(\alpha)|} (\|f\|_0^{m, \rho})^{\operatorname{Re}(\alpha) - k} \|f\|_{\nu_1}^{m, \rho} \dots \|f\|_{\nu_k}^{m, \rho},$$

which proves that  $f^\alpha \in S^{\operatorname{Re}(\alpha)m, \rho}(\mathbb{R}^n, \mathbb{C})$  as claimed.  $\square$

We come now to the approximation of symbols by compactly supported functions, which will be important for our subsequent construction of an oscillatory integral. As usual, balls in  $\mathbb{R}^n$  will be denoted  $B_r(0) := \{x \in \mathbb{R}^n : \|x\| \leq r\}$ .

**Proposition 2.10** *Let  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  be a compactly supported smooth function with*

$$\chi_{B_r(0)} = 1 \quad \text{and} \quad \operatorname{supp} \chi \subseteq B_R(0), \quad (2.31)$$

where  $0 < r < R$  and let  $\chi_\epsilon \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  be defined by  $\chi_\epsilon(x) = \chi(\epsilon x)$  for  $\epsilon > 0$ .

i.) One has  $\chi_\epsilon - 1 \in S^{0, \rho}(\mathbb{R}^n, \mathbb{C})$  for all  $\rho \in \mathbb{R}$ .

ii.) One has

$$\lim_{\epsilon \rightarrow 0} \chi_\epsilon = 1 \quad (2.32)$$

in the  $S^{m, \rho}$ -topology for all  $m > 0$  and  $\rho \leq 1$ .

iii.) For all  $F \in S^{m, \rho}(\mathbb{R}^n, V)$  we have

$$\lim_{\epsilon \rightarrow 0} \chi_\epsilon F = F \quad (2.33)$$

in the  $S^{m', \rho'}$ -topology for all  $m' > m$  and  $\rho' \leq \min(1, \rho)$ .

iv.) For all  $m' > m$  and  $\rho' \leq \min(1, \rho)$ , the compactly supported smooth functions  $\mathcal{C}_0^\infty(\mathbb{R}^n, V)$  are sequentially dense in  $S^{m, \rho}(\mathbb{R}^n, V)$  with respect to the  $S^{m', \rho'}$ -topology.

Proof. Clearly,  $1 \in S^{0, \rho}(\mathbb{R}^n, \mathbb{C})$  for all  $\rho \in \mathbb{R}$  and  $\chi_\epsilon \in \mathcal{C}_0^\infty(\mathbb{R}^n) \subseteq S^{m, \rho}(\mathbb{R}^n, \mathbb{C})$  for all  $m, \rho \in \mathbb{R}$  by Proposition 2.3, i.). For the second part, we clearly have pointwise convergence and even convergence in the  $\mathcal{C}^\infty$ -topology. For  $\mu = 0$  we have

$$\begin{aligned} \|\chi_\epsilon - 1\|_0^{m, \rho} &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}m} |\chi_\epsilon(x) - 1| \\ &= \sup_{\|x\| \geq \frac{r}{\epsilon}} (1 + \|x\|^2)^{-\frac{1}{2}m} |\chi_\epsilon(x) - 1| \\ &\leq c \sup_{\|x\| \geq \frac{r}{\epsilon}} (1 + \|x\|^2)^{-\frac{1}{2}m} \\ &= c \left( \frac{r^2 + \epsilon^2}{\epsilon^2} \right)^{-\frac{1}{2}m}, \end{aligned}$$

where  $c = \|\chi_\epsilon - 1\|_\infty < \infty$  by the compact support of  $\chi_\epsilon$ . This converges to zero since  $m > 0$ . For  $\mu \neq 0$  we have

$$\frac{\partial^{|\mu|} \chi_\epsilon}{\partial x^\mu}(x) = \epsilon^{|\mu|} \frac{\partial^{|\mu|} \chi}{\partial x^\mu}(\epsilon x)$$

and hence

$$\begin{aligned} \|\chi_\epsilon - 1\|_\mu^{m,\rho} &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \left| \frac{\partial^{|\mu|} \chi_\epsilon}{\partial x^\mu}(x) \right| \\ &\leq \sup_{\frac{r}{\epsilon} \leq \|x\| \leq \frac{R}{\epsilon}} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \epsilon^{|\mu|} c_\mu, \end{aligned}$$

where  $c_\mu = \|\partial_x^\mu \chi_\epsilon(x)\|_\infty < \infty$ , again thanks to the compact support. Now either  $m - \rho|\mu| \geq 0$ , then the supremum is taken at the smallest possible  $\|x\| = \frac{r}{\epsilon}$ , or  $m - \rho|\mu| < 0$ , then the supremum is taken at the largest possible  $\|x\| = \frac{R}{\epsilon}$ . Thus we get in the first case for  $\epsilon \leq 1$

$$\|\chi_\epsilon - 1\|_\mu^{m,\rho} \leq c_\mu \epsilon^{|\mu|} \left( \frac{r^2 + \epsilon^2}{\epsilon^2} \right)^{-\frac{1}{2}(m-\rho|\mu|)} \leq c'_\mu \epsilon^{m+(1-\rho)|\mu|},$$

and in the second case we get the same estimate with a different numerical constant  $c''_\mu$  instead of  $c_\mu$ . For the behaviour under  $\epsilon \rightarrow 0$  these factors do not play any role but the sign of  $m + (1 - \rho)|\mu|$  does: If  $\rho > 1$  then for large enough  $|\mu|$  we get divergence and hence  $\|\chi_\epsilon - 1\|_\mu^{m,\rho}$  does not converge to zero. If, on the other hand  $\rho \leq 1$ , then  $m + (1 - \rho)|\mu|$  is always strictly positive. In this case we have convergence  $\|\chi_\epsilon - 1\|_\mu^{m,\rho} \rightarrow 0$  for all  $\mu$ . This explains the condition  $\rho \leq 1$  and proves the second part. For the third part we rely on the estimates proved in Proposition 2.5, *i.*): for  $F \in S^{m,\rho}(\mathbb{R}^n, V)$  and a fixed seminorm  $q$  from the defining system  $\mathcal{Q}$  we get the estimate

$$\|(\chi_\epsilon - 1)F\|_{q,\mu}^{m',\rho'} \leq 2^{|\mu|} \max_{\nu \leq \mu} \|\chi_\epsilon - 1\|_\nu^{m,\rho} \max_{\nu' \leq \mu} \|F\|_{q,\nu}^{m,\rho}$$

for every  $m$  and  $m'$  provided  $m'(q) \geq m(q) + m$ , and every  $\rho$  and  $\rho'$  provided  $\rho'(q) \leq \min(\rho, \rho(q))$ . Now from the second part we know that  $\|\chi_\epsilon - 1\|_\nu^{m,\rho}$  converges to zero whenever  $\rho \leq 1$  and  $m > 0$ . This means that for the fixed seminorm  $q$  we get  $\|(\chi_\epsilon - 1)F\|_{q,\mu}^{m',\rho'} \rightarrow 0$  whenever  $m'(q) > m(q)$  and  $\rho'(q) \leq \min(1, \rho(q))$ . Since this is the condition for every  $q \in \mathcal{Q}$  we get the third part. Note that we are allowed to make the parameter  $m$  depend on  $q$  as long as we have  $m > 0$ . Thus  $m'(q) > m(q)$  does not have to be uniformly satisfied. The last part is now clear as it suffices to take  $\epsilon = \frac{1}{n}$  as usual.  $\square$

As the last operation to be discussed, we consider the restriction of a symbol to a subspace of its domain of definition. To this end, we take symbols  $F : \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \rightarrow V$  depending on two variables  $(x_1, x_2) \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ , and introduce the embeddings  $\iota_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ ,  $j = 1, 2$ , defined as  $\iota_1(x_1) := (x_1, 0)$  and  $\iota_2(x_2) := (0, x_2)$ . For a symbol  $F \in S^{m,\rho}(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, V)$ , we write

$$\iota_j^* F := F \circ \iota_j : \mathbb{R}^{n_j} \rightarrow V, \quad j = 1, 2. \quad (2.34)$$

**Lemma 2.11** *Let  $m, \rho$  be an order and a type for  $\mathcal{Q}$ . Then the restriction maps*

$$\iota_j^* : S^{m,\rho}(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, V) \longrightarrow S^{m,\rho}(\mathbb{R}^{n_j}, V) \quad (2.35)$$

*are linear and continuous,  $j = 1, 2$ .*

Proof. Let  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, V)$ . It is clear that  $\iota_j^* F$  is a smooth map from  $\mathbb{R}^{n_j}$  to  $V$ , and that  $\iota_j^*$  is linear. For  $j = 1$ , we estimate with  $\mathbf{q} \in \mathcal{Q}$ ,  $\mu \in \mathbb{N}_0^{n_1}$ ,

$$\begin{aligned} \|\iota_1^* F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} &= \sup_{x_1 \in \mathbb{R}^{n_1}} \frac{\mathbf{q}(\partial_{x_1}^\mu F(x_1, 0))}{(1 + \|(x_1, 0)\|^2)^{\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)}} \\ &\leq \sup_{x \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}} \frac{\mathbf{q}(\partial_x^{\mu \oplus 0} F(x))}{(1 + \|x\|^2)^{\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)}} \\ &= \|F\|_{\mathbf{q}, \mu \oplus 0}^{\mathbf{m}, \boldsymbol{\rho}}. \end{aligned}$$

This shows that  $\iota_1^* F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_1}, V)$ , and that  $\iota_1^*$  is continuous. The case  $j = 2$  is completely analogous.  $\square$

## 2.2 Symbol spaces

In this subsection, we introduce various spaces of symbols of arbitrary order and a vector-valued Schwartz space as suitable unions respectively intersections of the  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ . To show that these are intrinsic definitions, we will first discuss how our definition of the spaces  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  depends on the choice of the defining system of seminorms  $\mathcal{Q}$ . To this end, we shall proceed in two steps: First we show how one can pass from an arbitrary system to a filtrating one, then we compare two filtrating systems.

Now suppose  $\mathcal{Q}$  is an arbitrary defining system of continuous seminorms for  $V$ . Then we consider the larger system

$$\tilde{\mathcal{Q}} := \{\mathbf{q} = \max\{q_1, \dots, q_n\} \mid n \in \mathbb{N} \text{ and } q_1, \dots, q_n \in \mathcal{Q}\} \quad (2.36)$$

which is filtrating. Suppose now that  $\mathbf{m}$  is an order with respect to  $\mathcal{Q}$ . Then we want to extend  $\mathbf{m}$  to an order on  $\tilde{\mathcal{Q}}$  as follows. We define

$$\mathbf{m}_{\max}(\max\{q_1, \dots, q_n\}) := \max\{\mathbf{m}(q_1), \dots, \mathbf{m}(q_n)\}. \quad (2.37)$$

Clearly, this gives an order on  $\tilde{\mathcal{Q}}$  which extends  $\mathbf{m}$ . Analogously, for a type  $\boldsymbol{\rho}$  with respect to  $\mathcal{Q}$  we define a type  $\boldsymbol{\rho}_{\min}$  with respect to  $\tilde{\mathcal{Q}}$  extending  $\boldsymbol{\rho}$  by taking the minimum of the types  $\boldsymbol{\rho}(q_i)$  instead of the maximum.

**Proposition 2.12** *Let  $\mathcal{Q}$  be a defining system of continuous seminorms on  $V$  and  $\tilde{\mathcal{Q}}$  the corresponding filtrating system of finite maxima. Then for every order  $\mathbf{m}$  and every type  $\boldsymbol{\rho}$  with respect to  $\mathcal{Q}$  and their corresponding extensions  $\mathbf{m}_{\max}$  and  $\boldsymbol{\rho}_{\min}$  to  $\tilde{\mathcal{Q}}$  we have*

$$S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) = S^{\mathbf{m}_{\max}, \boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V) \quad (2.38)$$

as locally convex spaces.

Proof. First let  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  and let  $q_1, \dots, q_n \in \mathcal{Q}$  be given. We set  $\mathbf{q} := \max\{q_1, \dots, q_n\}$ . For  $\mu \in \mathbb{N}_0^n$  we have the estimate

$$\|F\|_{\mathbf{q}, \mu}^{\mathbf{m}_{\max}, \boldsymbol{\rho}_{\min}} = \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}_{\max}(\mathbf{q}) - \boldsymbol{\rho}_{\min}(\mathbf{q})|\mu|)} \mathbf{q}\left(\frac{\partial^{\mu} F}{\partial x^{\mu}}(x)\right)$$



$$\begin{aligned}
&\leq \sum_{i=1}^n \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}_i) - \boldsymbol{\rho}(\mathbf{q}_i)|\mu|)} \mathbf{q}_i \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\
&= \sum_{i=1}^n \|F\|_{\mathbf{q}_i, \mu}^{\mathbf{m}, \boldsymbol{\rho}}.
\end{aligned}$$

This shows  $F \in S^{\mathbf{m}_{\max}, \boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V)$  as well as the continuity of the inclusion map

$$S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m}_{\max}, \boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V).$$

Conversely, let  $F \in S^{\mathbf{m}_{\max}, \boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V)$  be given. Then  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  since all the seminorms  $\|\cdot\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}$  of the  $S^{\mathbf{m}, \boldsymbol{\rho}}$ -topology appear also as seminorms of the  $S^{\mathbf{m}_{\max}, \boldsymbol{\rho}_{\min}}$ -topology, since  $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$  and the order and type are extended to the larger system of seminorms. With respect to these seminorms  $\|\cdot\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}$ , the reverse inclusion

$$S^{\mathbf{m}_{\max}, \boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$$

is even isometric and hence continuous, too. Thus we have mutually inverse continuous inclusions proving the claim.  $\square$

Next we consider two defining systems of seminorms  $\mathcal{Q}$  and  $\mathcal{Q}'$  on  $V$  where we can assume that they are already filtrating. Thus for every  $\mathbf{q} \in \mathcal{Q}$  we find a  $\mathbf{q}' \in \mathcal{Q}'$  with  $\mathbf{q} \leq c \mathbf{q}'$  for some positive  $c > 0$ , and vice versa. In this situation we have the following statement:

**Proposition 2.13** *Let  $\mathcal{Q}$  and  $\mathcal{Q}'$  be defining systems of seminorms for  $V$  with  $\mathcal{Q}'$  being filtrating. Moreover, let  $\mathbf{m}, \mathbf{m}'$  be orders and  $\boldsymbol{\rho}, \boldsymbol{\rho}'$  be types for  $\mathcal{Q}, \mathcal{Q}'$ , respectively. If for every  $\mathbf{q} \in \mathcal{Q}$  there exists a  $\mathbf{q}' \in \mathcal{Q}'$  such that*

$$\mathbf{q} \leq c \mathbf{q}', \quad \mathbf{m}(\mathbf{q}) \geq \mathbf{m}'(\mathbf{q}'), \quad \text{and} \quad \boldsymbol{\rho}(\mathbf{q}) \leq \boldsymbol{\rho}'(\mathbf{q}'), \quad (2.39)$$

*then one has a continuous inclusion*

$$S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^n, V) \subseteq S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V). \quad (2.40)$$

Proof. Let  $\mathbf{q} \in \mathcal{Q}$  be given and choose  $\mathbf{q}'$  according to (2.39). Then for every  $\mu \in \mathbb{N}_0^n$  we have

$$\begin{aligned}
\|F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})|\mu|)} \mathbf{q} \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\
&\leq c \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}'(\mathbf{q}') - \boldsymbol{\rho}'(\mathbf{q}')|\mu|)} \mathbf{q}' \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\
&= c \|F\|_{\mathbf{q}', \mu}^{\mathbf{m}', \boldsymbol{\rho}'},
\end{aligned}$$

which shows the claim.  $\square$

**Corollary 2.14** *Let  $\mathcal{Q}$  and  $\mathcal{Q}'$  be defining systems of continuous seminorms on  $V$ . Moreover, let  $\mathbf{m}'$  and  $\boldsymbol{\rho}'$  be an order and a type for  $\mathcal{Q}'$ . Then there exists an order  $\mathbf{m}$  and a type  $\boldsymbol{\rho}$  for  $\mathcal{Q}$  such that*

$$S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^n, V) \subseteq S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \quad (2.41)$$

*is continuously included. If in addition  $\mathbf{m}'$  or  $\boldsymbol{\rho}'$  are bounded then  $\mathbf{m}$  and  $\boldsymbol{\rho}$  can be chosen to satisfy the same bounds, respectively.*

*Proof.* By Proposition 2.12 we can pass to a filtrating system without changing the symbol space on the left hand side. Thus we can assume that  $\mathcal{Q}'$  is already filtrating without restriction. Let  $q \in \mathcal{Q}$ , then we fix a particular choice  $q'(q) \in \mathcal{Q}'$  with  $q \leq c q'$  for some appropriate  $c > 0$ . This defines a map  $\mathcal{Q} \rightarrow \mathcal{Q}'$ , existing thanks to the fact that  $\mathcal{Q}'$  is filtrating. Now we define  $\mathbf{m}(q) := \mathbf{m}'(q'(q))$  and  $\boldsymbol{\rho}(q) := \boldsymbol{\rho}'(q'(q))$ . Then clearly the condition (2.39) from Proposition 2.13 is satisfied and we get (2.41). The statement about the bounds is then clear.  $\square$

**Corollary 2.15** *Let  $\mathcal{Q}$  and  $\mathcal{Q}'$  be two defining systems of seminorms for  $V$  and let  $F \in \mathcal{C}^\infty(\mathbb{R}^n, V)$  be a smooth function. Then  $F$  is a symbol of some (bounded) order  $\mathbf{m}$  and some (bounded) type  $\boldsymbol{\rho}$  for  $\mathcal{Q}$  iff  $F$  is a symbol of some (bounded) order  $\mathbf{m}'$  and some (bounded) type  $\boldsymbol{\rho}'$  for  $\mathcal{Q}'$  (and the same bounds).*

*Proof.* By Proposition 2.12 we can assume to have filtrating systems from the beginning. Since the extension of the order and the type according to that Proposition clearly preserves the bounds, Corollary 2.14 can be applied in both directions.  $\square$

We can now use the last corollaries to speak about the space of *all* symbols: there are two alternatives whether or not we allow for bounded orders only:

**Definition 2.16** *Let  $V$  be a sequentially complete locally convex space. Then we define for a given type  $\boldsymbol{\rho}$  for a defining system of seminorms  $\mathcal{Q}$*

$$S^{\infty, \boldsymbol{\rho}}(\mathbb{R}^n, V) := \bigcup_{\mathbf{m} \text{ bounded}} S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \quad (2.42)$$

and

$$S^{\infty+, \boldsymbol{\rho}}(\mathbb{R}^n, V) := \bigcup_{\mathbf{m}} S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V). \quad (2.43)$$

Moreover, we set

$$S^\infty(\mathbb{R}^n, V) := S^{\infty, 1}(\mathbb{R}^n, V), \quad S^{\infty+}(\mathbb{R}^n, V) := S^{\infty+, 1}(\mathbb{R}^n, V) \quad (2.44)$$

and

$$\underline{S}(\mathbb{R}^n, V) := \bigcup_{-1 < \boldsymbol{\rho} \leq 1} S^{\infty+, \boldsymbol{\rho}}(\mathbb{R}^n, V). \quad (2.45)$$

It follows that for another defining system of seminorms  $\mathcal{Q}'$  we get the same spaces  $S^\infty(\mathbb{R}^n, V)$ ,  $S^{\infty+}(\mathbb{R}^n, V)$ ,  $\underline{S}(\mathbb{R}^n, V)$ , which are therefore intrinsically defined. The space  $\underline{S}(\mathbb{R}^n, V)$  will be relevant in the context of oscillatory integrals, presented in Section 3.

Note that for a Banach space  $V$  with  $\mathcal{Q}$  consisting just of the norm itself we have  $S^{\infty, \rho}(\mathbb{R}^n, V) = S^{\infty+, \rho}(\mathbb{R}^n, V)$  for all types  $\rho \in \mathbb{R}$ . However, in general we have a proper inclusion

$$S^{\infty, \rho}(\mathbb{R}^n, V) \subset S^{\infty+, \rho}(\mathbb{R}^n, V). \quad (2.46)$$

Also the intersection of all the symbol spaces is of interest: here we get an analog of the usual Schwartz space. First we note the following simple facts:

**Lemma 2.17** *Let  $V$  be a sequentially complete locally convex space and  $F \in \mathcal{C}^\infty(\mathbb{R}^n, V)$ . Then the following statements are equivalent:*

*i.) For all continuous seminorms  $q$  of a defining system  $\mathcal{Q}$ , for all  $\mu \in \mathbb{N}_0^n$  and all  $m \in \mathbb{N}_0$  one has*

$$q_{m, \mu}(F) = \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{\frac{m}{2}} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) < \infty. \quad (2.47)$$

*ii.) For all orders  $\mathbf{m}$  and all types  $\rho$  for a given defining system  $\mathcal{Q}$  of continuous seminorms one has  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$ .*

*iii.) For all orders  $\mathbf{m}$  and one type  $\rho$  for a given defining system  $\mathcal{Q}$  of continuous seminorms one has  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$ .*

Proof. First we note that if *i.)* holds for a defining system of seminorms  $\mathcal{Q}$  then it also holds for *all* continuous seminorms of  $V$ . This is clear. Thus assume *i.)* and let  $\mathcal{Q}$  be a defining system of seminorms. Moreover, fix an order  $\mathbf{m}$  and a type  $\rho$  for this system  $\mathcal{Q}$ . Then for  $\mu \in \mathbb{N}_0^n$  we have

$$\|F\|_{q, \mu}^{\mathbf{m}, \rho} = \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(q) - \rho(q)|\mu|)} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \leq q_{m, \mu}(F),$$

with  $m$  being any integer larger than  $-\mathbf{m}(q) + \rho(q)|\mu|$ . This shows that  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$ . The implication *ii.)*  $\Rightarrow$  *iii.)* is trivial. Thus assume *iii.)* and hence let  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$  for all orders  $\mathbf{m}$  and a fixed type  $\rho$ . Then let  $m \in \mathbb{N}_0$  and  $\mu \in \mathbb{N}_0^n$  be given. We have

$$\begin{aligned} q_{m, \mu}(F) &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{\frac{m}{2}} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\ &\leq \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(q) - \rho(q)|\mu|)} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\ &= \|F\|_{q, \mu}^{\mathbf{m}, \rho}, \end{aligned}$$

where we have to choose an order  $\mathbf{m}$  such that  $\mathbf{m}(q) - \rho(q)|\mu| \leq -m$ . This is clearly possible as we can e.g. take the constant order with  $\mathbf{m} = -m + \rho(q)|\mu|$ . Thus *i.)* follows.  $\square$

Thus for the intersection of all symbol spaces the type  $\rho$  does not play any role any more. Also the dependence on the chosen system of seminorms  $\mathcal{Q}$  disappears. This motivates the following definition:

**Definition 2.18 (Vector-valued Schwartz space)** *Let  $V$  be a sequentially complete locally convex space. Then we define the symbols of order  $-\infty$  by*

$$S^{-\infty}(\mathbb{R}^n, V) := \bigcap_{\mathbf{m}, \boldsymbol{\rho}} S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V). \quad (2.48)$$

*We also use the notation*

$$\mathcal{S}(\mathbb{R}^n, V) := S^{-\infty}(\mathbb{R}^n, V) \quad (2.49)$$

*and call  $\mathcal{S}(\mathbb{R}^n, V)$  the space of  $V$ -valued Schwartz functions.*

Clearly, the  $V$ -valued Schwartz functions are a straightforward generalization of the scalar case. The Schwartz space  $\mathcal{S}(\mathbb{R}^n, V)$  will always be endowed with the locally convex topology determined by the seminorms  $q_{\mathbf{m}, \boldsymbol{\mu}}$  as in (2.47). We call this the  $S^{-\infty}$ - or the Schwartz topology of  $\mathcal{S}(\mathbb{R}^n, V)$ . We collect now some easy properties of the Schwartz space:

**Proposition 2.19** *Let  $V$  be a sequentially complete locally convex space with a defining system of seminorms  $\mathcal{Q}$ .*

- i.) The Schwartz space  $\mathcal{S}(\mathbb{R}^n, V)$  is sequentially complete and even complete when  $V$  is complete.*
- ii.) We have continuous inclusions*

$$\mathcal{C}_0^\infty(\mathbb{R}^n, V) \longrightarrow \mathcal{S}(\mathbb{R}^n, V) \longrightarrow S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \quad (2.50)$$

*for all orders  $\mathbf{m}$  and all types  $\boldsymbol{\rho}$  for  $\mathcal{Q}$ .*

- iii.)  $\mathcal{C}_0^\infty(\mathbb{R}^n, V)$  is sequentially dense in  $\mathcal{S}(\mathbb{R}^n, V)$  and  $\mathcal{S}(\mathbb{R}^n, V)$  is sequentially dense in  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  in the  $S^{\mathbf{m}', \boldsymbol{\rho}'}$ -topology whenever  $\mathbf{m}' > \mathbf{m}$  and  $\boldsymbol{\rho}' \leq \min(1, \boldsymbol{\rho})$ .*
- iv.) The partial derivatives are continuous linear maps*

$$\frac{\partial^{|\nu|}}{\partial x^\nu} : \mathcal{S}(\mathbb{R}^n, V) \longrightarrow \mathcal{S}(\mathbb{R}^n, V) \quad (2.51)$$

*satisfying the estimate (equality)*

$$q_{\mathbf{m}, \boldsymbol{\mu}} \left( \frac{\partial^{|\nu|}}{\partial x^\nu} F \right) = q_{\mathbf{m}, \boldsymbol{\mu} + \nu}(F). \quad (2.52)$$

- v.) If  $W$  and  $U$  are two other sequentially complete locally convex spaces and  $\mu : V \times W \longrightarrow U$  is a continuous bilinear map then it induces continuous bilinear maps*

$$\mu : \mathcal{S}(\mathbb{R}^n, V) \times \mathcal{S}(\mathbb{R}^n, W) \longrightarrow \mathcal{S}(\mathbb{R}^n, U), \quad (2.53)$$

$$\mu : S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V) \times \mathcal{S}(\mathbb{R}^n, W) \longrightarrow \mathcal{S}(\mathbb{R}^n, U), \quad (2.54)$$

*and*

$$\mu : \mathcal{S}(\mathbb{R}^n, V) \times S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^n, W) \longrightarrow \mathcal{S}(\mathbb{R}^n, U) \quad (2.55)$$

*for all orders  $\mathbf{m}$  and types  $\boldsymbol{\rho}$  for  $\mathcal{Q}$  and all orders  $\mathbf{m}'$  and types  $\boldsymbol{\rho}'$  for some defining system of seminorms  $\mathcal{Q}'$  for  $W$ .*

vi.) For all orders  $m \in \mathbb{R}$  and types  $\rho \in \mathbb{R}$  the pointwise multiplication

$$S^{m,\rho}(\mathbb{R}^n, \mathbb{C}) \times \mathcal{S}(\mathbb{R}^n, V) \longrightarrow \mathcal{S}(\mathbb{R}^n, V) \quad (2.56)$$

is a continuous bilinear map.

Proof. The first statement can most easily be checked using the explicit seminorms  $q_{m,\mu}$  in the same spirit as the proof of Proposition 2.3, *iii.*). Then also the second part is clear since we get a continuous inclusion of  $\mathcal{C}_K^\infty(\mathbb{R}^n, V)$  into  $\mathcal{S}(\mathbb{R}^n, V)$  with estimates like

$$q_{m,\mu}(F) = \|F\|_{q,\mu}^{m,0} \leq c_K p_{K,|\mu|,q}(F)$$

as in the proof of Proposition 2.3, *i.*), for every compact subset  $K \subseteq \mathbb{R}^n$ . The second inclusion is continuous thanks to the estimate  $\|F\|_{q,\mu}^{m,\rho}(F) \leq q_{m,\mu}(F)$  for  $m$  an integer being at least  $-\mathbf{m}(q) + \rho(q)|\mu|$ , which we have established in the proof of Lemma 2.17. The density of  $\mathcal{C}_0^\infty(\mathbb{R}^n, V)$  in  $\mathcal{S}(\mathbb{R}^n, V)$  is checked directly as in the scalar case. The second statement of part *iii.*) is clear as  $\mathcal{C}_0^\infty(\mathbb{R}^n, V)$  has this property by Proposition 2.10, *iv.*). The fourth part is clear since the estimate (2.52) is obvious by definition. For part *v.*) we first consider the case (2.54). Thus let  $F \in S^{m,\rho}(\mathbb{R}^n, V)$  and  $G \in \mathcal{S}(\mathbb{R}^n, W)$  be given. The continuity of  $\mu$  means that for a continuous seminorm  $r$  on  $U$  we find  $q \in \mathcal{Q}$  and a continuous seminorm  $q'$  on  $W$  such that

$$r(\mu(v, w)) \leq q(v) q'(w)$$

for all  $v \in V$  and  $w \in W$  since we can assume that the defining system  $\mathcal{Q}$  on  $V$  is already filtrating by Proposition 2.12. Then for  $m \in \mathbb{N}_0$  and  $\kappa \in \mathbb{N}_0^n$  we estimate using the Leibniz rule

$$\begin{aligned} r_{m,\kappa}(\mu(F, G)) &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{\frac{m}{2}} r \left( \frac{\partial^{|\kappa|} \mu(F, G)}{\partial x^\kappa}(x) \right) \\ &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{\frac{m}{2}} r \left( \sum_{\nu + \nu' = \kappa} \binom{\kappa}{\nu} \mu \left( \frac{\partial^{|\nu|} F}{\partial x^\nu}(x), \frac{\partial^{|\nu'|} G}{\partial x^{\nu'}}(x) \right) \right) \\ &\leq 2^{|\kappa|} \sup_{x \in \mathbb{R}^n} \sum_{\nu + \nu' = \kappa} (1 + \|x\|^2)^{\frac{m}{2}} q \left( \frac{\partial^{|\nu|} F}{\partial x^\nu}(x) \right) q' \left( \frac{\partial^{|\nu'|} G}{\partial x^{\nu'}}(x) \right) \\ &= 2^{|\kappa|} \sup_{x \in \mathbb{R}^n} \sum_{\nu + \nu' = \kappa} (1 + \|x\|^2)^{-\frac{1}{2}(\mathbf{m}(q) - \rho(q)|\nu|)} q \left( \frac{\partial^{|\nu|} F}{\partial x^\nu}(x) \right) \\ &\quad (1 + \|x\|^2)^{\frac{m}{2} + \frac{1}{2}(\mathbf{m}(q) - \rho(q)|\nu|)} q' \left( \frac{\partial^{|\nu'|} G}{\partial x^{\nu'}}(x) \right) \\ &\leq 2^{|\kappa|} \sum_{\nu + \nu' = \kappa} \|F\|_{q,\nu}^{m,\rho} q'_{m',\nu'}(G), \end{aligned}$$

with  $m' = m + \mathbf{m}(q) - \rho(q)|\nu|$ . This shows the continuity of (2.54) and (2.55) is analogous. But then (2.53) follows as well since  $\mathcal{S}(\mathbb{R}^n, V) \longrightarrow S^{m,\rho}(\mathbb{R}^n, V)$  is continuous.

In fact, one can also estimate the bilinear expression directly. The last part is then clear.  $\square$

**Corollary 2.20** *Let  $\mathcal{A}$  be a (sequentially) complete locally convex algebra and let  $\mathcal{M}$  be a (sequentially) complete locally convex topological module over  $\mathcal{A}$ . Then  $\mathcal{S}(\mathbb{R}^n, \mathcal{A})$  is a (sequentially) complete locally convex algebra and  $\mathcal{S}(\mathbb{R}^n, \mathcal{M})$  is a (sequentially) complete locally convex topological module over  $\mathcal{S}(\mathbb{R}^n, \mathcal{A})$ .*

### 2.3 Affine symmetries and symbol-valued symbols

In this subsection we investigate the action of the affine group of  $\mathbb{R}^n$  on the symbol spaces. For  $A \in \mathrm{GL}_n(\mathbb{R})$  and a translation  $y \in \mathbb{R}^n$ , we denote their pullback by  $(A^*F)(x) = F(Ax)$  and  $(\tau_y^*F)(x) = F(x+y)$  as usual. We start with the following basic observations:

**Lemma 2.21** *Let  $q$  be a continuous seminorm on  $V$  and  $m, \rho \in \mathbb{R}$ . Then for  $F \in \mathcal{C}^\infty(\mathbb{R}^n, V)$  we have for all  $\mu \in \mathbb{N}_0^n$  and all  $A \in \mathrm{GL}_n(\mathbb{R})$*

$$\|A^*F\|_{q,\mu}^{m,\rho} \leq c_\mu^{m,\rho}(A) \sum_{\substack{\nu \in \mathbb{N}_0^n \\ |\nu|=|\mu|}} \|F\|_{q,\nu}^{m,\rho} \quad (2.57)$$

with some  $c_\mu^{m,\rho}(A) > 0$  depending continuously on  $A$  and satisfying  $c_\mu^{m,\rho}(\mathbb{1}) = 1$ .

Proof. As usual, this is to be understood as an inequality in  $[0, +\infty]$ . First we note that with the operator norm of  $A$  we have

$$q\left(\frac{\partial^{|\mu|}(A^*F)}{\partial x^\mu}(x)\right) \leq \|A\|^{|\mu|} \sum_{\substack{\nu \in \mathbb{N}_0^n \\ |\nu|=|\mu|}} q\left(\frac{\partial^{|\nu|}F}{\partial x^\nu}(Ax)\right)$$

for all  $x \in \mathbb{R}^n$  by the chain rule. Next we recall that

$$\frac{1}{\|A\|} \|y\| \leq \|A^{-1}y\| \leq \|A^{-1}\| \|y\|$$

for an invertible  $A \in \mathrm{GL}_n(\mathbb{R})$ . We have to distinguish a few cases. Depending on the sign of  $m - \rho|\mu|$  we first get

$$\left(1 + \|A^{-1}y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} \leq \begin{cases} \left(1 + \|A^{-1}\|^2 \|y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| < 0 \\ \left(1 + \|A\|^{-2} \|y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| \geq 0. \end{cases}$$

In the case  $\|A^{-1}\| \leq 1$  and hence  $\|A\| \geq 1$  we can continue the estimate by

$$\left(1 + \|A^{-1}y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} \leq \begin{cases} \left(1 + \|y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| < 0 \\ \|A\|^{m-\rho|\mu|} \left(1 + \|y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| \geq 0. \end{cases} \quad (*)$$

Conversely, in the case  $\|A^{-1}\| \geq 1$  we get

$$\left(1 + \|A^{-1}y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} \leq \|A^{-1}\|^{-m+\rho|\mu|} (1 + \|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \quad (**)$$

for the case  $m - \rho|\mu| < 0$ . For  $m - \rho|\mu| \geq 0$  we still have to distinguish the two possibilities  $\|A\| \leq 1$  and  $\|A\| \geq 1$ . For  $\|A\| \leq 1$  we have

$$\left(1 + \|A^{-1}y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} \leq (1 + \|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)}, \quad (*)$$

and in the case  $\|A\| > 1$  we finally get

$$\left(1 + \|A^{-1}y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} \leq \|A\|^{m-\rho|\mu|} (1 + \|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)}. \quad (**)$$

Combining the four cases (\*), (\*\*), (\*), and (\*\*) we get the estimate

$$\|A\|^{|\mu|} \left(1 + \|A^{-1}y\|^2\right)^{-\frac{1}{2}(m-\rho|\mu|)} \leq c_\mu^{m,\rho}(A) (1 + \|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)}$$

where

$$c_\mu^{m,\rho}(A) = \|A\|^{|\mu|} \max \left\{ 1, \|A\|^{m-\rho|\mu|}, \|A^{-1}\|^{-m+\rho|\mu|} \right\}.$$

Since  $A \mapsto A^{-1}$  is continuous and since the operator norm is continuous as well this constant depends continuously on  $A$  and clearly satisfies  $c_\mu^{m,\rho}(\mathbb{1}) = 1$ . It is now easy to see that we get the estimate (2.57).  $\square$

**Lemma 2.22** *Let  $q$  be a continuous seminorm on  $V$  and  $m, \rho \in \mathbb{R}$ . Then for  $F \in \mathcal{C}^\infty(\mathbb{R}^n, V)$  we have for all  $\mu \in \mathbb{N}_0^n$  and all  $y \in \mathbb{R}^n$*

$$\|\tau_y^* F\|_{q,\mu}^{m,\rho} \leq c_\mu^{m,\rho}(y) \|F\|_{q,\mu}^{m,\rho}, \quad (2.58)$$

with some positive  $c_\mu^{m,\rho}(y) > 0$  being a scalar symbol  $c_\mu^{m,\rho} \in S^{|m-\rho|\mu|,1}(\mathbb{R}^n)$ .

Proof. We proceed similar as in the previous lemma. First it is clear that

$$q \left( \frac{\partial^{|\mu|}(\tau_y^* F)}{\partial x^\mu}(x) \right) = q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x + y) \right) \quad (*)$$

by the chain rule. For the prefactor in the definition of the seminorm we first consider the following elementary estimate: There is a constant  $c > 0$  such that for all  $x, y \geq 0$  we have

$$\frac{1}{1 + (x - y)^2} \leq c \frac{1 + y^2}{1 + x^2}. \quad (**)$$

We now use this inequality to consider first the case  $m - \rho|\mu| \geq 0$ . There we have

$$(1 + \|x - y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \leq \frac{1}{(1 + \|\|x\| - \|y\|\|^2)^{\frac{1}{2}(m-\rho|\mu|)}}$$



$$\stackrel{(**)}{\leq} c^{\frac{1}{2}(m-\rho|\mu|)} \left( \frac{1 + \|y\|^2}{1 + \|x\|^2} \right)^{\frac{1}{2}(m-\rho|\mu|)}.$$

This gives the estimate

$$\begin{aligned} \|\tau_y^* F\|_{q,\mu}^{m,\rho} &\stackrel{(*)}{=} \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x+y) \right) \\ &= \sup_{x \in \mathbb{R}^n} (1 + \|x-y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\ &\leq c^{\frac{1}{2}(m-\rho|\mu|)} \sup_{x \in \mathbb{R}^n} \left( \frac{1 + \|y\|^2}{1 + \|x\|^2} \right)^{\frac{1}{2}(m-\rho|\mu|)} q \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\ &= c^{\frac{1}{2}(m-\rho|\mu|)} (1 + \|y\|^2)^{\frac{1}{2}(m-\rho|\mu|)} \|F\|_{q,\mu}^{m,\rho}. \end{aligned}$$

The case  $m - \rho|\mu| < 0$  is even simpler. Here we have  $-\frac{1}{2}(m - \rho|\mu|) \geq 0$  and hence

$$\begin{aligned} (1 + \|x-y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} &\leq (1 + 2\|x\|^2 + 2\|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \\ &\leq 2^{-\frac{1}{2}(m-\rho|\mu|)} (1 + \|x\|^2 + \|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \\ &\leq 2^{-\frac{1}{2}(m-\rho|\mu|)} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} (1 + \|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)}. \end{aligned}$$

By an analogous argument as for the previous case this results in the estimate

$$\|\tau_y^* F\|_{q,\mu}^{m,\rho} \leq 2^{-\frac{1}{2}(m-\rho|\mu|)} (1 + \|y\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \|F\|_{q,\mu}^{m,\rho}.$$

So for each  $m, \rho \in \mathbb{R}$ ,  $\mu \in \mathbb{N}_0^n$ , we find a constant  $C_\mu^{m,\rho} > 0$  such that  $\|\tau_y^* F\|_{q,\mu}^{m,\rho} \leq C_\mu^{m,\rho} (1 + \|y\|^2)^{\frac{1}{2}|m-\rho|\mu|} \|F\|_{q,\mu}^{m,\rho}$  for all  $y \in \mathbb{R}^n$ . As the function  $y \mapsto 1 + \|y\|^2$  is clearly in  $S^{2,1}(\mathbb{R}^n, \mathbb{C})$ , an application of Lemma 2.9 yields  $y \mapsto (1 + \|y\|^2)^{\frac{1}{2}|m-\rho|\mu|} \in S^{|m-\rho|\mu|,1}(\mathbb{R}^n, \mathbb{C})$ , and the proof is finished.  $\square$

**Remark 2.23** Note that for the translations  $\tau_y$  the pre-factor  $c_\mu^{m,\rho}(y)$  in (2.58) is always a symbol of *non-negative* order, even if  $m - \rho|\mu|$  was negative. Thus the bounds in (2.58) typically *grow* with  $y$  and are also growing with increasing differentiations  $\mu$  unless  $\rho = 0$ .

As an easy consequence of the two lemmas we get the affine invariance of the symbol spaces:

**Proposition 2.24** *Let  $V$  be a sequentially complete locally convex space and  $\mathcal{Q}$  a defining system of seminorms. Moreover, let  $\mathbf{m}$  and  $\boldsymbol{\rho}$  be an order and a type for  $\mathcal{Q}$ . Then the affine group  $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$  of  $\mathbb{R}^n$  acts on the symbols  $S^{\mathbf{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$  via pull-backs by continuous endomorphisms.*

Proof. The pull-backs with  $A \in \text{GL}_n(\mathbb{R})$  or with a translation by  $y \in \mathbb{R}^n$  map  $S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$  continuously into itself according to Lemma 2.21 and Lemma 2.22, respectively. The fact that this gives a (right) group action is clear.  $\square$

In a next step we want to refine this statement for the translations: we want to show that the map  $y \mapsto \tau_y^* F$  is actually smooth. We begin with the following observation:

**Lemma 2.25** *Let  $F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V)$ . Then the map*

$$\mathbb{R}^n \ni y \mapsto \tau_y^* F \in S^{\mathbf{m}, \rho}(\mathbb{R}^n, V) \quad (2.59)$$

*is continuous at zero provided  $\rho \geq 0$ .*

Proof. We have to show that  $\tau_y^* F \rightarrow F$  in the  $S^{\mathbf{m}, \rho}$ -topology for  $y \rightarrow 0$ . Let  $x \in \mathbb{R}^n$  be given. Then we have for  $y \in \mathbb{R}^n$  and  $\mathbf{q} \in \mathcal{Q}$  by virtue of the mean value theorem

$$\begin{aligned} \mathbf{q} \left( \frac{\partial^{|\mu|}(\tau_y^* F)}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) &= \mathbf{q} \left( \int_0^1 \left( \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu} \right) (x + ty) \, dty^i \right) \\ &\leq \sqrt{n} \sup_{\substack{t \in [0,1] \\ i=1, \dots, n}} \mathbf{q} \left( \left( \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu} \right) (x + ty) \right) \|y\|. \quad (*) \end{aligned}$$

Now we use the fact that  $F$  is a symbol. This means that the  $(\mu + e_i)$ -th derivative of  $F$  satisfies

$$\mathbf{q} \left( \left( \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu} \right) (x + ty) \right) \leq (1 + \|x + ty\|^2)^{\frac{1}{2}(m - \rho|\mu| - \rho)} \|F\|_{\mathbf{q}, \mu + e_i}^{\mathbf{m}, \rho}, \quad (**)$$

where for abbreviation we have set  $m = \mathbf{m}(\mathbf{q})$  and  $\rho = \rho(\mathbf{q})$ . Then we get

$$\begin{aligned} &\|\tau_y^* F - F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \rho} \\ &= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(m - \rho|\mu|)} \mathbf{q} \left( \frac{\partial^{|\mu|}(\tau_y^* F)}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\ &\stackrel{(*), (**)}{\leq} \sqrt{n} \|y\| \sup_{\substack{x \in \mathbb{R}^n \\ t \in [0,1] \\ i=1, \dots, n}} (1 + \|x\|^2)^{-\frac{1}{2}(m - \rho|\mu|)} (1 + \|x + ty\|^2)^{\frac{1}{2}(m - \rho|\mu| - \rho)} \|F\|_{\mathbf{q}, \mu + e_i}^{\mathbf{m}, \rho} \\ &= \sqrt{n} \|y\| \sup_{\substack{x \in \mathbb{R}^n \\ t \in [0,1] \\ i=1, \dots, n}} (1 + \|x - ty\|^2)^{-\frac{1}{2}(m - \rho|\mu|)} (1 + \|x\|^2)^{\frac{1}{2}(m - \rho|\mu| - \rho)} \|F\|_{\mathbf{q}, \mu + e_i}^{\mathbf{m}, \rho}. \end{aligned}$$

We can again estimate the first factor by the same techniques as in the Lemma 2.22: we get a constant  $c$  (depending on  $m$ ,  $\rho$ , and  $\mu$  but not on  $ty$  or  $x$ ) such that we can continue our estimate and get

$$\|\tau_y^* F - F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \rho}$$

$$\begin{aligned}
&\leq c \|y\| \sup_{\substack{x \in \mathbb{R}^n \\ t \in [0,1] \\ i=1,\dots,n}} (1 + \|ty\|^2)^{\frac{1}{2}|m-\rho|\mu|} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)+\frac{1}{2}(m-\rho|\mu|-\rho)} \|F\|_{\mathbf{q},\mu+e_i}^{\mathbf{m},\rho} \\
&\leq c \|y\| (1 + \|y\|^2)^{\frac{1}{2}|m-\rho|\mu|} \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{\rho}{2}} \max_{i=1,\dots,n} \|F\|_{\mathbf{q},\mu+e_i}^{\mathbf{m},\rho}.
\end{aligned}$$

Now if  $\rho \geq 0$  then the supremum over all  $x \in \mathbb{R}^n$  exists and hence we get an estimate of the form

$$\|\tau_y^* F - F\|_{\mathbf{q},\mu}^{\mathbf{m},\rho} \leq c' \|y\| (1 + \|y\|^2)^{\frac{1}{2}|m-\rho|\mu|},$$

from which the continuity at  $y = 0$  follows.  $\square$

**Lemma 2.26** *Let  $F \in S^{\mathbf{m},\rho}(\mathbb{R}^n, V)$  be given with  $\rho \geq 0$ . Then we have*

$$\lim_{\epsilon \rightarrow 0} \frac{\tau_{\epsilon e_i}^* F - F}{\epsilon} = \frac{\partial F}{\partial x^i} \quad (2.60)$$

in the  $S^{\mathbf{m},\rho}$ -topology for all  $i = 1, \dots, n$ .

Proof. We proceed analogously to the continuity statement. Let again  $\mathbf{q} \in \mathcal{Q}$  and set  $m = \mathbf{m}(\mathbf{q})$  and  $\rho = \rho(\mathbf{q})$  for abbreviation. First we note that for all  $x \in \mathbb{R}^n$  and  $\mu \in \mathbb{N}_0^n$  we have by repeated use of the mean value theorem

$$\begin{aligned}
&\mathbf{q} \left( \frac{1}{\epsilon} \left( \frac{\partial^{|\mu|}(\tau_{\epsilon e_i}^* F)}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) - \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\
&= \mathbf{q} \left( \frac{1}{\epsilon} \left( \frac{\partial^{|\mu|} F}{\partial x^\mu}(x + \epsilon e_i) - \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) - \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\
&= \mathbf{q} \left( \int_0^1 \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu}(x + t\epsilon e_i) dt - \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) \\
&= \mathbf{q} \left( \int_0^1 \int_0^1 \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu}(x + t s \epsilon e_i) t \epsilon ds dt \right) \\
&\leq \epsilon \sup_{s \in [0,1]} \mathbf{q} \left( \frac{\partial^2}{\partial (x^i)^2} \frac{\partial^{|\mu|} F}{\partial x^\mu}(x + s \epsilon e_i) \right) \\
&\leq \epsilon \sup_{s \in [0,1]} (1 + \|x + s \epsilon e_i\|^2)^{\frac{1}{2}(m-\rho|\mu|-2\rho)} \|F\|_{\mathbf{q},\mu+2e_i}^{\mathbf{m},\rho},
\end{aligned}$$

since by assumption  $F \in S^{\mathbf{m},\rho}(\mathbb{R}^n, V)$ . Thus we get

$$\begin{aligned}
&\left\| \frac{1}{\epsilon} (\tau_{\epsilon e_i}^* F - F) - \frac{\partial F}{\partial x^i} \right\|_{\mathbf{q},\mu}^{\mathbf{m},\rho} \\
&= \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} \mathbf{q} \left( \frac{1}{\epsilon} \left( \frac{\partial^{|\mu|}(\tau_{\epsilon e_i}^* F)}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right) - \frac{\partial}{\partial x^i} \frac{\partial^{|\mu|} F}{\partial x^\mu}(x) \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon \sup_{\substack{x \in \mathbb{R}^n \\ s \in [0,1]}} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} (1 + \|x + s\epsilon e_i\|^2)^{\frac{1}{2}(m-\rho|\mu|-2\rho)} \|F\|_{q,\mu+2e_i}^{\mathbf{m},\rho} \\
&= \epsilon \sup_{\substack{x \in \mathbb{R}^n \\ s \in [0,1]}} (1 + \|x - s\epsilon e_i\|^2)^{-\frac{1}{2}(m-\rho|\mu|)} (1 + \|x\|^2)^{\frac{1}{2}(m-\rho|\mu|-2\rho)} \|F\|_{q,\mu+2e_i}^{\mathbf{m},\rho} \\
&\leq c\epsilon \sup_{\substack{x \in \mathbb{R}^n \\ s \in [0,1]}} (1 + \|s\epsilon e_i\|^2)^{\frac{1}{2}|m-\rho|\mu|} (1 + \|x\|^2)^{-\frac{1}{2}(m-\rho|\mu|)+\frac{1}{2}(m-\rho|\mu|-2\rho)} \|F\|_{q,\mu+2e_i}^{\mathbf{m},\rho} \\
&= c\epsilon(1 + \epsilon^2)^{\frac{1}{2}|m-\rho|\mu|} \sup_{x \in \mathbb{R}^n} (1 + \|x\|^2)^{-\rho} \|F\|_{q,\mu+2e_i}^{\mathbf{m},\rho}.
\end{aligned}$$

Using again  $\rho \geq 0$  shows that the remaining supremum is finite. Thanks to the pre-factor  $\epsilon$  we get the desired limit (2.60).  $\square$

These two lemmas are now enough to conclude the following smoothness statement of the action of the translations:

**Proposition 2.27** *Let  $V$  be a sequentially complete locally convex space and let  $\mathcal{Q}$  be a defining system of seminorms for  $V$ . Let  $\mathbf{m}$  and  $\rho$  be an order and a type for  $\mathcal{Q}$  and assume  $\rho \geq 0$ . Then the action  $\tau$  of  $\mathbb{R}^n$  on  $S^{\mathbf{m},\rho}(\mathbb{R}^n, V)$  by translations is smooth, i.e. for every  $F \in S^{\mathbf{m},\rho}(\mathbb{R}^n, V)$  the map  $\tau(F): y \mapsto \tau_y^* F$  is a smooth map. The derivatives are explicitly given by*

$$\frac{\partial^{|\mu|}}{\partial y^\mu} \tau_y^* F = \tau_y^* \frac{\partial^{|\mu|} F}{\partial x^\mu}. \quad (2.61)$$

Proof. This is a general argument about group actions of Lie groups: We know already that  $\tau(F)$  is continuous at  $y = 0$  by Lemma 2.25. Moreover, every map  $\tau_y^*$  is continuous by Lemma 2.22. Thus we have by the group action property

$$\lim_{y \rightarrow y'} \tau_y^* F = \lim_{y \rightarrow 0} \tau_{y'+y}^* F = \lim_{y \rightarrow 0} \tau_y^* \tau_{y'}^* F = \tau_{y'}^* F$$

in the  $S^{\mathbf{m},\rho}$ -topology since we have continuity at zero. This shows continuity everywhere. Moreover, by the same argument

$$\lim_{\epsilon \rightarrow 0} \frac{\tau_{y+\epsilon e_i}^* F - \tau_y^* F}{\epsilon} = \lim_{\epsilon \rightarrow 0} \tau_y^* \frac{\tau_{\epsilon e_i}^* F - F}{\epsilon} = \tau_y^* \lim_{\epsilon \rightarrow 0} \frac{\tau_{\epsilon e_i}^* F - F}{\epsilon} = \tau_y^* \frac{\partial F}{\partial x^i},$$

using Lemma 2.26 and the continuity of  $\tau_y^*$ . This shows that  $\tau(F)$  has first partial derivatives everywhere given as in (2.61). Now  $\frac{\partial F}{\partial x^i} \in S^{\mathbf{m}-\rho,\rho}(\mathbb{R}^n, V) \subseteq S^{\mathbf{m},\rho}(\mathbb{R}^n, V)$  thanks to  $\rho \geq 0$  and Proposition 2.3, *iv.*) as well as Proposition 2.4. Thus the partial derivatives  $\frac{\partial}{\partial y^i} \tau(F) = \tau(\frac{\partial F}{\partial x^i})$  are again of the form as we started with. Hence they are continuous and thus  $\tau(F)$  is  $\mathcal{C}^1$ . This allows to iterate the above argument finishing the proof.  $\square$

As a first application of the affine invariance of the spaces  $S^{\mathbf{m},\rho}(\mathbb{R}^n, V)$  we get the following generalization of the approximation from Proposition 2.10, *iii.*)

**Corollary 2.28** *Let  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  satisfy  $\chi|_{B_r(0)} = 1$  for some  $r > 0$ . Consider  $\tau_y^* \chi_\epsilon$  for  $\epsilon > 0$  and  $y \in \mathbb{R}^n$  where as usual  $\chi_\epsilon(x) = \chi(\epsilon x)$ . Then for every  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  we have*

$$\lim_{\epsilon \rightarrow 0} (\tau_y^* \chi_\epsilon) F = F \quad (2.62)$$

*in the  $S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^n, V)$ -topology provided  $\boldsymbol{\rho}' \leq \min(1, \boldsymbol{\rho})$  and  $\mathbf{m}' > \mathbf{m}$ .*

Proof. We have  $(\tau_y^* \chi_\epsilon) F = \tau_y^* (\chi_\epsilon \tau_{-y} F)$  and then the continuity of  $\tau_y^*$  according to Proposition 2.24 allows to exchange  $\tau_y^*$  with the limit. Then the result follows from Proposition 2.10, *iii.*)  $\square$

The smoothness of the translations also allows to consider symbols taking values in other symbol spaces, as we shall do now. Recall that given an order  $\mathbf{m}$  and a type  $\boldsymbol{\rho}$  for a defining system of seminorms  $\mathcal{Q}$  on  $V$ , the symbol space  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  is a sequentially complete locally convex space (Proposition 2.3, *iii.*), which can therefore be used as a target space instead of  $V$ . To define symbols taking values in it, we first have to specify an order  $\hat{\mathbf{m}}$  and a type  $\hat{\boldsymbol{\rho}}$  on the seminorms  $\|\cdot\|_{\mathbf{q}, \mu}^{\mathbf{m}(\mathbf{q}), \boldsymbol{\rho}(\mathbf{q})}$  generating the topology of  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ . For  $\mathbf{q} \in \mathcal{Q}$  and  $\mu \in \mathbb{N}_0^n$ , we put

$$\hat{\mathbf{m}}(\|\cdot\|_{\mathbf{q}, \mu}^{\mathbf{m}(\mathbf{q}), \boldsymbol{\rho}(\mathbf{q})}) := \max\{0, \mathbf{m}(\mathbf{q})\}, \quad \hat{\boldsymbol{\rho}}(\|\cdot\|_{\mathbf{q}, \mu}^{\mathbf{m}(\mathbf{q}), \boldsymbol{\rho}(\mathbf{q})}) := \boldsymbol{\rho}(\mathbf{q}). \quad (2.63)$$

**Proposition 2.29** *Let  $V$  be a sequentially complete locally convex space with defining system of seminorms  $\mathcal{Q}$ , and let  $\mathbf{m}$  be an order and  $\boldsymbol{\rho} \geq 0$  a positive type for  $\mathcal{Q}$ . Moreover, let  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, V)$  be given. Then*

$$F_1: \mathbb{R}^{n_1} \rightarrow S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_2}, V), \quad F_1(x_1): x_2 \mapsto F(x_1, x_2) \quad (2.64)$$

*is a symbol in  $S^{\hat{\mathbf{m}}, \hat{\boldsymbol{\rho}}}(\mathbb{R}^{n_1}, S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_2}, V))$  of order  $\hat{\mathbf{m}}$  and type  $\hat{\boldsymbol{\rho}}$  (2.63), and the map*

$$S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, V) \longrightarrow S^{\hat{\mathbf{m}}, \hat{\boldsymbol{\rho}}}(\mathbb{R}^{n_1}, S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_2}, V)) \quad (2.65)$$

$$F \longmapsto F_1 \quad (2.66)$$

*is linear and continuous. Explicitly, one has the bound*

$$\|F_1\|_{\|\cdot\|_{\mathbf{q}, \mu}^{\hat{\mathbf{m}}, \hat{\boldsymbol{\rho}}}, \nu} \leq \|F\|_{\mathbf{q}, \nu \oplus \mu}^{\mathbf{m}, \boldsymbol{\rho}} \quad (2.67)$$

*for  $\mathbf{q} \in \mathcal{Q}$ ,  $\nu \in \mathbb{N}_0^{n_1}$ ,  $\mu \in \mathbb{N}_0^{n_2}$ . Completely analogous statements hold for the map*

$$F_2: \mathbb{R}^{n_2} \rightarrow S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_1}, V), \quad F_2(x_2): x_1 \mapsto F(x_1, x_2). \quad (2.68)$$

Proof. In terms of the embedding  $\iota_2: \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ ,  $\iota_2(x_2) := (0, x_2)$ , and the previously discussed translations  $\tau$ , the map  $F_1$  reads  $F_1(x_1) = \iota_2^*(\tau_{x_1 \oplus 0}^* F)$ . But according to Proposition 2.27,  $x_1 \mapsto \tau_{x_1 \oplus 0}^* F$  is a smooth map from  $\mathbb{R}^{n_1}$  to  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, V)$ , and according to Lemma 2.11,  $\iota_2^*: S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, V) \rightarrow S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_2}, V)$  is linear and continuous. Hence  $F_1: \mathbb{R}^{n_1} \rightarrow S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{n_2}, V)$  is smooth. Since  $F \mapsto F_1$  is clearly linear,

it only remains to verify the estimate (2.67). To this end, let  $q \in \mathcal{Q}$ ,  $\nu \in \mathbb{N}_0^{n_1}$ ,  $\mu \in \mathbb{N}_0^{n_2}$ , and put  $\hat{q} := \|\cdot\|_{q,\mu}^{m,\rho}$  for short. The seminorm in question is

$$\begin{aligned} \|F_1\|_{\hat{q},\nu}^{m,\hat{\rho}} &= \sup_{x_1 \in \mathbb{R}^{n_1}} \frac{\|\partial_{x_1}^\nu F_1(x_1)\|_{q,\mu}^{m,\rho}}{(1 + \|x_1\|^2)^{\frac{1}{2}(\hat{m}(\hat{q}) - \hat{\rho}(\hat{q})|\nu|)}} \\ &= \sup_{x_1 \in \mathbb{R}^{n_1}} \frac{1}{(1 + \|x_1\|^2)^{\frac{1}{2}(\hat{m}(\hat{q}) - \hat{\rho}(\hat{q})|\nu|)}} \sup_{x_2 \in \mathbb{R}^{n_2}} \frac{q(\partial_{x_2}^\mu \partial_{x_1}^\nu F_1(x_1)(x_2))}{(1 + \|x_2\|^2)^{\frac{1}{2}(m(q) - \rho(q)|\mu|)}}. \end{aligned}$$

Note that by definition of  $\hat{m}$ ,  $\hat{\rho}$ , the powers  $k := -\frac{1}{2}(\hat{m}(\hat{q}) - \rho(q)|\nu|)$  and  $k' := -\frac{1}{2}(m(q) - \rho(q)|\mu|)$  satisfy  $\max\{k, k', k + k'\} \leq K$  with  $K := -\frac{1}{2}(m(q) - \rho(q)|\nu \oplus \mu|)$  for all  $\mu, \nu$ . Hence we can use the inequality (2.23) to estimate

$$\|F_1\|_{\hat{q},\nu}^{m,\hat{\rho}} \leq \sup_{\substack{x_1 \in \mathbb{R}^{n_1} \\ x_2 \in \mathbb{R}^{n_2}}} \frac{q(\partial^{\nu \oplus \mu} F(x_1, x_2))}{(1 + \|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}(m(q) - \rho(q)|\nu \oplus \mu|)}} = \|F\|_{q,\nu \oplus \mu}^{m,\rho}.$$

This establishes (2.67) and thus in particular the continuity of  $F \mapsto F_1$ . The arguments for  $F_2$  are completely analogous.  $\square$

### 3 Oscillatory integrals for vector-valued symbols

#### 3.1 Construction of the integral map

We now come to the definition of oscillatory integrals of symbols. Again, we proceed in close analogy to the scalar case, see [21, Sect. 7.8] as well as [20]. The essential idea is to use the Riemann integral for compactly supported smooth functions and show that it enjoys a remarkable continuity property with respect to the symbol topologies. We are here not interested in the most general case, where oscillatory integrals are used to define maps from test function spaces to distributions, as discussed in [21, Sect. 7.8]. Instead we are just interested in the values of the oscillatory integrals per se. To this end, we endow  $\mathbb{R}^n$  with a non-degenerate bilinear form  $\langle \cdot, \cdot \rangle$ . Then we consider for  $F \in \mathcal{C}_0(\mathbb{R}^n, V)$  the integral with an oscillatory phase

$$I_0(F) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} F(x, p), \quad (3.1)$$

which is a well-defined Riemann integral thanks to the continuity of the integrand and the compact support of  $F$ . The integral defines a linear map

$$I_0: \mathcal{C}_0(\mathbb{R}^{2n}, V) \longrightarrow V, \quad (3.2)$$

which is continuous in the  $\mathcal{C}_0$ -topology. Since the  $\mathcal{C}_0$ -topology is coarser than every  $\mathcal{C}_0^k$ -topology for  $k \in \mathbb{N}_0 \cup \{+\infty\}$ , we see that for all  $k$  we have a continuous map

$$I_0: \mathcal{C}_0^k(\mathbb{R}^{2n}, V) \longrightarrow V. \quad (3.3)$$

Up to now, we have not used any particular properties of the phase function besides its continuity. However, it turns out that the continuity with respect to the  $\mathcal{C}_0^k$ -topologies is not the right one to extend  $I_0$  to the symbol spaces.

Instead we have to show the continuity of  $I_0$  with respect to some appropriate  $S^{m,\rho}$ -topology, and this step will make use of more specific properties of the phase function. We begin with the following preparations. Consider the polynomial

$$P(x) := (i + x_1) \cdots (i + x_n) \quad (3.4)$$

on  $\mathbb{R}^n$  which is clearly of degree  $n$  and hence a scalar symbol  $P \in S^{n,1}(\mathbb{R}^n, \mathbb{C})$ .

Since each factor  $(i + x_k)$  is non-vanishing, we can define arbitrary powers  $(i + x_k)^s$  for  $s \in \mathbb{Z}$ , which are symbols of order  $s$ . For  $s \geq 0$ , this follows from Lemma 2.9, and for  $s < 0$  by explicit differentiation. Note that given a symbol  $F \in S^{m,\rho}(\mathbb{R}^{2n}, V)$  of some order  $m$  and type  $\rho \leq 1$ , the function

$$\mathbb{R}^{2n} \ni (x, p) \mapsto P^s(x)P^s(p)F(x, p) \in V \quad (3.5)$$

is a symbol of order  $m + 2sn$  and type  $\rho$ . This follows directly by application of Corollary 2.7, *i.*, since  $(x, p) \mapsto P^s(x)P^s(p)$  is of order  $2sn$  and type 1, and  $\rho \leq 1$  by assumption.

We also note the well-known fact that given  $s \in \mathbb{N}_0$ , there exists a differential operator

$$Q_s = \sum_{|\mu|, |\nu| \leq s} a_s^{\mu\nu} \frac{\partial^{|\mu|}}{\partial x^\mu} \frac{\partial^{|\nu|}}{\partial p^\nu} \quad (3.6)$$

with constant coefficients  $a_s^{\mu\nu} \in \mathbb{C}$  such that

$$Q_s e^{i\langle p, x \rangle} = P^s(x)P^s(p) e^{i\langle p, x \rangle}. \quad (3.7)$$

After these preparatory remarks, we now derive the crucial estimate of the integral  $I_0$  with respect to the symbol topologies. The proof is based on the usual technique of converting differentiability properties of the integrand to damping factors by integration by parts against  $e^{i\langle p, x \rangle}$ . As shown below, for this technique to work we only have to make a restriction on the type, but not on the order.

**Lemma 3.1** *Let  $\mathcal{Q}$  be a defining system of seminorms for  $V$ , with order  $m$  and type  $\rho$  such that  $-1 < \rho \leq 1$ . Then for every  $q \in \mathcal{Q}$  there exists a constant  $c > 0$  and  $N \in \mathbb{N}_0$  such that for all  $F \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$  we have*

$$q(I_0(F)) \leq c \sum_{|\mu| \leq N} \|F\|_{q,\mu}^{m,\rho}. \quad (3.8)$$

*Proof.* Let  $F \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$  have compact support in a compact interval  $K \subseteq \mathbb{R}^{2n}$ . Then we compute using (3.7)

$$I_0(F) = \frac{1}{(2\pi)^n} \int_K e^{i\langle p, x \rangle} F(x, p) \, d^n x \, d^n p$$



$$\begin{aligned}
&= \frac{1}{(2\pi)^n} \int_K \frac{1}{P^s(x)P^s(p)} (Q_s e^{i\langle p, x \rangle}) F(x, p) d^n x d^n p \\
&= \frac{1}{(2\pi)^n} \int_K e^{i\langle p, x \rangle} Q_s^T \frac{F(x, p)}{P^s(x)P^s(p)} d^n x d^n p,
\end{aligned} \tag{*}$$

where  $Q_s^T = \sum_{0 \leq |\mu|, |\nu| \leq s} (-1)^{|\mu|+|\nu|} a_s^{\mu\nu} \frac{\partial^{|\mu|}}{\partial x^\mu} \frac{\partial^{|\nu|}}{\partial p^\nu}$  denotes the transposed differential operator and  $s \in \mathbb{N}_0$  is arbitrary. Indeed, the integration by parts is possible since  $F$  has compact support inside the interval  $K$ . Since (\*) is valid for all  $s \in \mathbb{N}_0$ , the idea is to use a large enough  $s$  which produces under the integral an integrable symbol on the right hand side, independent of  $K$ . Since  $F$  has compact support it is a symbol for any order and any type. Thus also the function  $(x, p) \mapsto \frac{F(x, p)}{P^s(x)P^s(p)}$  is a symbol, say of order  $\mathbf{m} - 2sn$  and type  $\boldsymbol{\rho}$ . Thus for all  $\mu, \nu \in \mathbb{N}_0^n$  we have the estimate

$$q \left( \frac{\partial^{|\mu|}}{\partial x^\mu} \frac{\partial^{|\nu|}}{\partial p^\nu} \frac{F(x, p)}{P^s(x)P^s(p)} \right) \leq (1 + \|(x, p)\|^2)^{\frac{1}{2}(\mathbf{m}(q) - 2sn - \boldsymbol{\rho}(q)|\mu \oplus \nu|)} \left\| \frac{F(\cdot, \cdot)}{P^s(\cdot)P^s(\cdot)} \right\|_{q, \mu \oplus \nu}^{\mathbf{m} - 2sn, \boldsymbol{\rho}}$$

for all  $s \in \mathbb{N}_0$ . We know that  $|\mu \oplus \nu| = |\mu| + |\nu| \leq 2sn$  as the operator  $Q_s$  is of order  $2sn$  only. Hence the condition  $\boldsymbol{\rho}(q) > -1$  shows that there is a  $s \in \mathbb{N}_0$  such that for all  $|\mu|, |\nu| \leq sn$  we have

$$\mathbf{m}(q) - 2sn - \boldsymbol{\rho}(q)|\mu \oplus \nu| < -2(n+1). \tag{3.9}$$

In fact, we get the left hand side as negative as we want by taking large enough  $s$ . Finally, by Proposition 2.5, *i.*), we get the estimate

$$\left\| \frac{F}{P^s(\cdot)P^s(\cdot)} \right\|_{q, \mu \oplus \nu}^{\mathbf{m} - 2sn, \boldsymbol{\rho}} \leq 2^{|\mu|+|\nu|} \max_{\mu' \oplus \nu' \leq \mu \oplus \nu} \left\| \frac{1}{P^s(\cdot)P^s(\cdot)} \right\|_{\mu' \oplus \nu'}^{-2sn, 1} \max_{\mu'' \oplus \nu'' \leq \mu \oplus \nu} \|F\|_{q, \mu'' \oplus \nu''}^{\mathbf{m}, \boldsymbol{\rho}},$$

since we have  $\boldsymbol{\rho} \leq 1$  and  $\frac{1}{P^s(\cdot)P^s(\cdot)}$  is a symbol of order  $-2sn$  and type 1 according to Lemma 2.9. Taking now  $s$  large enough so that (3.9) is satisfied we get the estimate

$$\begin{aligned}
q(I_0(F)) &= q \left( \frac{1}{(2\pi)^n} \int_K e^{i\langle p, x \rangle} Q_s^T \frac{F(x, p)}{P^s(x)P^s(p)} d^n x d^n p \right) \\
&\leq \frac{1}{(2\pi)^n} \int_K q \left( \sum_{0 \leq \mu, \nu \leq s} a_s^{\mu\nu} (-1)^{|\mu|+|\nu|} \frac{\partial^{|\mu|}}{\partial x^\mu} \frac{\partial^{|\nu|}}{\partial p^\nu} \frac{F(x, p)}{P^s(x)P^s(p)} \right) d^n x d^n p \\
&\leq \frac{1}{(2\pi)^n} \sum_{0 \leq \mu, \nu \leq s} |a_s^{\mu\nu}| \int_K (1 + \|(x, p)\|^2)^{-(n+1)} d^n x d^n p \left\| \frac{F}{P^s(\cdot)P^s(\cdot)} \right\|_{q, \mu \oplus \nu}^{\mathbf{m} - 2sn, \boldsymbol{\rho}} \\
&\leq c \sum_{0 \leq |\mu|, |\nu| \leq s} \|F\|_{q, \mu \oplus \nu}^{\mathbf{m}, \boldsymbol{\rho}},
\end{aligned}$$

with the constant

$$c = \frac{2^{2sn}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \frac{d^n x d^n p}{(1 + \|(x, p)\|^2)^{-(n+1)}} \max_{0 \leq |\mu|, |\nu| \leq s} |a_s^{\mu\nu}| \max_{0 \leq |\mu'|, |\nu'| \leq s} \left\| \frac{1}{P^s(\cdot)P^s(\cdot)} \right\|_{\mu' \oplus \nu'}^{-2sn, 1} < \infty.$$

Note that the integral is finite indeed as we were able to make the exponent (3.9) negative enough such that the dependence on the compact interval  $K$  disappears.  $\square$

We now define oscillatory integrals for symbols  $F \in S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V)$  of non-compact support by extending the integral  $I_0$  defined on  $\mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$ . Doing so, we will rely in an essential manner on the preceding lemma and Proposition 2.10, *iv.*), and therefore restrict to types  $\rho$  with  $-1 < \rho \leq 1$ . The order  $\mathbf{m}$  will be arbitrary.

To describe the extension procedure, we consider in addition to  $\mathbf{m}$  and  $-1 < \rho \leq 1$  an auxiliary order  $\mathbf{m}' > \mathbf{m}$  and type  $-1 < \rho' \leq \rho$  for  $\mathcal{Q}$ , and the corresponding inclusions

$$I_0: \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V) \subset S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V) \subset S^{\mathbf{m}',\rho'}(\mathbb{R}^{2n}, V) \longrightarrow V.$$

In general,  $\mathcal{C}_0^\infty(\mathbb{R}^{2n}, V) \subset S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V)$  is not (sequentially) dense in the  $S^{\mathbf{m},\rho}$ -topology. But according to Proposition 2.10, *iv.*), the sequential closure of  $\mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$  in the weaker  $S^{\mathbf{m}',\rho'}$ -topology contains  $S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V)$ . Moreover, according to the bound (3.8),  $I_0: \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V) \longrightarrow V$  is a continuous linear map in the  $S^{\mathbf{m}',\rho'}$ -topology. We can thus extend  $I_0$  to a continuous linear map from the sequential completion of  $\mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$  in the  $S^{\mathbf{m}',\rho'}$ -topology to  $V$ . The restriction of this extension to  $S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V)$  is our definition of oscillatory integral on  $S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V)$ ; it is denoted by

$$I_{\mathbf{m}',\rho'}^{\mathbf{m},\rho}: S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V) \longrightarrow V. \quad (3.10)$$

**Theorem 3.2** *Let  $V$  be a sequentially complete locally convex space with defining system of seminorms  $\mathcal{Q}$ , and  $\mathbf{m}$ ,  $-1 < \rho \leq 1$  an order and a type for  $\mathcal{Q}$ .*

- i.) The integrals  $I_{\mathbf{m}',\rho'}^{\mathbf{m},\rho} := I_{\mathbf{m}',\rho'}^{\mathbf{m},\rho}$  (3.10) are independent of the order and type  $\mathbf{m}'$ ,  $\rho'$  as long as  $\mathbf{m}' > \mathbf{m}$  and  $-1 < \rho' \leq \rho$ .*
- ii.)  $I_{\mathbf{m},\rho}: S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V) \longrightarrow V$  is linear and continuous.*
- iii.) For  $F \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$ , we have  $I_{\mathbf{m},\rho}(F) = I_0(F)$ .*
- iv.) For orders  $\mathbf{m}, \mathbf{m}'$ , types  $-1 < \rho, \rho' \leq 1$ , and  $F \in S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V) \cap S^{\mathbf{m}',\rho'}(\mathbb{R}^{2n}, V)$ , we have*

$$I_{\mathbf{m},\rho}(F) = I_{\mathbf{m}',\rho'}^{\mathbf{m},\rho}(F). \quad (3.11)$$

*Proof.* For the first part, let  $\mathbf{m}', \mathbf{m}''$  be orders and  $\rho', \rho''$  types for  $\mathcal{Q}$ , with  $\mathbf{m}', \mathbf{m}'' > \mathbf{m}$  and  $-1 < \rho', \rho'' \leq \rho$ , and  $F \in S^{\mathbf{m},\rho}(\mathbb{R}^{2n}, V)$ . We have to show  $I_{\mathbf{m}',\rho'}^{\mathbf{m},\rho}(F) = I_{\mathbf{m}'',\rho''}^{\mathbf{m},\rho}(F)$ . By the above construction of these maps, there exist sequences  $\{F'_n\}, \{F''_n\} \subset \mathcal{C}_0^\infty$  converging to  $F$  in the topology of  $S^{\mathbf{m}',\rho'}(\mathbb{R}^{2n}, V)$  and  $S^{\mathbf{m}'',\rho''}(\mathbb{R}^{2n}, V)$ , respectively, and

$$I_{\mathbf{m}',\rho'}^{\mathbf{m},\rho}(F) = \lim_{n \rightarrow \infty} I_0(F'_n), \quad I_{\mathbf{m}'',\rho''}^{\mathbf{m},\rho}(F) = \lim_{n \rightarrow \infty} I_0(F''_n). \quad (3.12)$$

To show that these limits coincide, let  $\mathbf{m}''', \rho'''$  be an order and type with  $\mathbf{m}''' \geq \mathbf{m}', \mathbf{m}''$ , and  $-1 < \rho''' \leq \rho', \rho''$ . Fixing a seminorm  $q \in \mathcal{Q}$ , we can use the bound (3.8) and (2.10) to estimate with some constants  $c > 0$ ,  $N \in \mathbb{N}_0$ ,

$$q(I_0(F'_n) - I_0(F''_n)) \leq c \sum_{|\mu| \leq N} \|F'_n - F''_n\|_{q,\mu}^{\mathbf{m}''',\rho'''}$$

$$\begin{aligned}
&\leq c \sum_{|\mu| \leq N} \left( \|F'_n - F\|_{q,\mu}^{\mathbf{m}''', \boldsymbol{\rho}'''} + \|F - F''_n\|_{q,\mu}^{\mathbf{m}''', \boldsymbol{\rho}'''} \right) \\
&\leq c \sum_{|\mu| \leq N} \left( \|F'_n - F\|_{q,\mu}^{\mathbf{m}', \boldsymbol{\rho}'} + \|F - F''_n\|_{q,\mu}^{\mathbf{m}'', \boldsymbol{\rho}''} \right).
\end{aligned}$$

In view of the approximation properties of the sequences  $\{F'_n\}$ ,  $\{F''_n\}$ , the last expression converges to zero for  $n \rightarrow \infty$ , *i.e.*, we have  $q(I_0(F'_n) - I_0(F''_n)) \rightarrow 0$ . Since  $q$  was arbitrary, (3.12) now gives  $I_{\mathbf{m}', \boldsymbol{\rho}'}^{\mathbf{m}, \boldsymbol{\rho}}(F) = I_{\mathbf{m}'', \boldsymbol{\rho}''}^{\mathbf{m}, \boldsymbol{\rho}}(F)$ . From now on, we write  $I^{\mathbf{m}, \boldsymbol{\rho}} := I_{\mathbf{m}', \boldsymbol{\rho}'}^{\mathbf{m}, \boldsymbol{\rho}}$  for this integral. For part *ii.*), by construction,  $I^{\mathbf{m}, \boldsymbol{\rho}}: S^{\mathbf{m}, \boldsymbol{\rho}} \rightarrow V$  is a linear map which is continuous in the  $S^{\mathbf{m}', \boldsymbol{\rho}'}$ -topology for  $\mathbf{m}' > \mathbf{m}$  and  $-1 < \boldsymbol{\rho}' \leq \boldsymbol{\rho}$ . But as the topology of  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$  is stronger than that of  $S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^{2n}, V)$ , this map is continuous in the  $S^{\mathbf{m}, \boldsymbol{\rho}}$ -topology as well. By the very definition of  $I^{\mathbf{m}, \boldsymbol{\rho}}$ , we have  $I^{\mathbf{m}, \boldsymbol{\rho}}(F) = I_0(F)$  for  $F \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$ , *i.e.* *iii.*) holds. It remains to check *iv.*), and to this end, we consider an order  $\mathbf{m}'' > \mathbf{m}, \mathbf{m}'$  and type  $-1 < \boldsymbol{\rho}'' \leq \boldsymbol{\rho}, \boldsymbol{\rho}'$ . Then for any  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V) \cap S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^{2n}, V)$ , there exists a sequence  $\{F_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$  converging to  $F$  in the topology of  $S^{\mathbf{m}'', \boldsymbol{\rho}''}(\mathbb{R}^{2n}, V)$ , and in view of *i.*), we have

$$I^{\mathbf{m}, \boldsymbol{\rho}}(F) = I_{\mathbf{m}'', \boldsymbol{\rho}''}^{\mathbf{m}, \boldsymbol{\rho}}(F) = \lim_{n \rightarrow \infty} I_0(F_n) = I_{\mathbf{m}'', \boldsymbol{\rho}''}^{\mathbf{m}', \boldsymbol{\rho}'}(F) = I^{\mathbf{m}', \boldsymbol{\rho}'}(F). \quad (3.13)$$

This proves (3.11).  $\square$

The compatibility (3.11) of the integral maps  $I^{\mathbf{m}, \boldsymbol{\rho}}$  with the structure of the symbol spaces allows us to consistently define an oscillatory integral on the space  $\underline{S}(\mathbb{R}^{2n}, V)$ , see (2.45), consisting of all  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$ , with arbitrary orders  $\mathbf{m}$  and types  $-1 < \boldsymbol{\rho} \leq 1$ .

**Definition 3.3** *The oscillatory integral is the linear map  $I: \underline{S}(\mathbb{R}^{2n}, V) \rightarrow V$  uniquely determined by  $I|_{S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)} := I^{\mathbf{m}, \boldsymbol{\rho}}$ ,  $-1 < \boldsymbol{\rho} \leq 1$ . If the target space or the domain of integration needs to be emphasized, we write more precisely  $I_V$  or  $I_{\mathbb{R}^{2n}, V}$  instead of  $I$ . We also use the symbolic notation*

$$(2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} F(p, x) := I(F). \quad (3.14)$$

Note that according to the discussion in Section 2.3, the space  $\underline{S}(\mathbb{R}^{2n}, V)$  and the oscillatory integral  $I$  do not depend on a choice of defining system  $\mathcal{Q}$  of seminorms, but are intrinsically defined.

### 3.2 Calculational rules for the oscillatory integral

We now derive the main properties of the integral  $I$ . To begin with, we note how oscillatory integrals can be computed in practice.

**Proposition 3.4** *i.) Let  $F \in \underline{S}(\mathbb{R}^{2n}, V)$ ,  $p_0, x_0 \in \mathbb{R}^n$ , and  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, \mathbb{R})$  with  $\chi(p, x) = 1$  for  $(p, x)$  in some open neighborhood of  $(0, 0)$ . Then the oscillatory integral of  $F$  is the limit of Riemann integrals*

$$I(F) = (2\pi)^{-n} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \chi(\varepsilon(p - p_0), \varepsilon(x - x_0)) F(p, x). \quad (3.15)$$

ii.) Let  $\mathcal{Q}$  be a defining system of seminorms on  $V$ , with order  $\mathbf{m}$  and type  $\boldsymbol{\rho}$  such that there exist constants  $C_1, C_2 \in \mathbb{R}$  satisfying

$$\mathbf{m}(\mathbf{q}) \leq C_1, \quad 1 \geq \boldsymbol{\rho}(\mathbf{q}) \geq C_2 > -1 \quad (3.16)$$

for all  $\mathbf{q} \in \mathcal{Q}$ . Then there exists  $s \in \mathbb{N}$ , and  $b_{\mu\nu} \in \mathbb{C}$ ,  $\mu, \nu \in \mathbb{N}_0^n$ ,  $|\mu|, |\nu| \leq s$ , such that for all  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$ , the oscillatory integral is given by a convergent Riemann integral

$$I(F) = \sum_{|\mu|, |\nu| \leq s} b_{\mu\nu} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \frac{\partial^{|\mu|}}{\partial p^\mu} \frac{\partial^{|\nu|}}{\partial x^\nu} \left( \frac{F(p, x)}{\prod_{k=1}^n (i + p_k)^s (i + x_k)^s} \right). \quad (3.17)$$

Proof. Fixing a defining system of seminorms  $\mathcal{Q}$  on  $V$ , we consider a symbol  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$  for some order  $\mathbf{m}$  and type  $-1 < \boldsymbol{\rho} \leq 1$  for  $\mathcal{Q}$ . Furthermore, let  $\mathbf{m}', \boldsymbol{\rho}'$  be an auxiliary order and type for  $\mathcal{Q}$  such that  $\mathbf{m}' > \mathbf{m}$  and  $-1 < \boldsymbol{\rho}' \leq \boldsymbol{\rho}$ . It has been shown in Corollary 2.28 that  $(\chi_\varepsilon F)(p, x) := \chi(\varepsilon(p - p_0), \varepsilon(x - x_0))F(p, x)$  converges to  $F$  in the topology of  $S^{\mathbf{m}', \boldsymbol{\rho}'}(\mathbb{R}^{2n}, V)$  as  $\varepsilon \rightarrow 0$ . Since  $\chi_\varepsilon F \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$ , the formula  $I(F) = \lim_{\varepsilon \rightarrow 0} I_0(\chi_\varepsilon F)$  (3.15) holds by definition of  $I$  as the  $S^{\mathbf{m}, \boldsymbol{\rho}}$ -continuous extension of  $I_0$ . This proves the first part. The second part is basically a corollary of the proof of Lemma 3.1: One first checks that if (3.16) holds, then there exists  $s \in \mathbb{N}_0$  such that the inequality (3.9) is valid for all  $\mathbf{q} \in \mathcal{Q}$  for the same value of  $s$ . Using a cutoff function  $\chi$  as in the first part of this proposition, we can then apply the arguments in the proof of Lemma 3.1 to  $\chi_\varepsilon F \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$  to conclude that there exist coefficients  $b_{\mu\nu}$  such that

$$I_0(\chi_\varepsilon F) = \sum_{|\mu|, |\nu| \leq s} b_{\mu\nu} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \frac{\partial^{|\mu|}}{\partial p^\mu} \frac{\partial^{|\nu|}}{\partial x^\nu} \frac{\chi(\varepsilon p, \varepsilon x) F(p, x)}{\prod_{k=1}^n (i + p_k)^s (i + x_k)^s}. \quad (3.18)$$

To control the limit  $\varepsilon \rightarrow 0$ , we again use the same arguments as in Lemma 3.1: For any seminorm  $\mathbf{q}$ , we find an integrable upper bound to  $(p, x) \mapsto \mathbf{q}(\partial_p^\mu \partial_x^\nu P^{-s}(p) P^{-s}(x) F(p, x))$ . This allows us to carry out the limit  $\varepsilon \rightarrow 0$  in (3.18). Namely, applying Leibniz' rule, we see that all terms in (3.18) which contain derivatives of  $\chi$ , and hence factors of  $\varepsilon$ , converge to zero as  $\varepsilon \rightarrow 0$  because the derivatives of  $F$  and the damping factors are bounded in each seminorm  $\mathbf{q}$ . Only the term with no derivatives on  $\chi$  remains, and as this has an integrable upper bound, and  $\chi(0, 0) = 1$ , we obtain the claimed formula (3.17) for  $I(F) = \lim_{\varepsilon \rightarrow 0} I_0(\chi_\varepsilon F)$ .  $\square$

If  $V$  is a Banach space and  $\mathcal{Q}$  consists of just its norm, then (3.16) is clearly satisfied for any order  $m$ , and any admissible type  $-1 < \rho \leq 1$ . So in this case, the oscillatory integrals can always be reformulated as improper Riemann integrals. But if  $\mathcal{Q}$  is infinite, and  $\mathbf{m}$  unbounded, this is no longer the case. Nonetheless, also in this general situation, oscillatory integrals exhibit many of the familiar properties of Riemann integrals. In particular, they are compatible with continuous linear maps, and the usual rules of substitution and integration by parts still apply, as we now show in the following lemmas and propositions.

**Lemma 3.5** *Let  $V, U$  be sequentially complete locally convex spaces,  $A: V \rightarrow U$  a continuous linear map, and  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ . Then  $AF: (p, x) \mapsto AF(p, x)$  is a symbol in  $\underline{\mathbb{S}}(\mathbb{R}^{2n}, U)$ , and*

$$A I_V(F) = I_U(AF). \quad (3.19)$$

Proof. First we note that the equation holds for compactly supported  $F$ . But then the usual continuity and approximation argument shows that the equation also holds for arbitrary  $F$ .  $\square$

**Remark 3.6** If we consider an *antilinear* continuous map  $C: V \rightarrow U$  instead, the only difference to the above described situation is that the oscillating factor  $e^{i\langle p, x \rangle}$  has to be conjugated. This conjugation can be compensated by a variable substitution  $p \rightarrow -p$  in the integrals. So in this case, we have,  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ ,

$$C I_V(F) = I_U(C F_-) \quad \text{with} \quad F_-(p, x) := F(-p, x). \quad (3.20)$$

**Lemma 3.7** *Let  $q, y \in \mathbb{R}^n$ ,  $A \in GL(n, \mathbb{R})$ , and denote by  $A^T$  the transpose of  $A$  with respect to the chosen inner product on  $\mathbb{R}^n$ . Then, for any  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ , the functions*

$$F_{q,y,A}(p, x) := e^{-i\langle p, y \rangle} F(Ap + q, x), \quad (3.21)$$

$$F^{q,y,A}(p, x) := e^{-i\langle q, x \rangle} F(p, A^T x + y), \quad (3.22)$$

are elements of  $\underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$  as well, and

$$I(F_{q,y,A}) = I(F^{q,y,A}). \quad (3.23)$$

Proof. For  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ , an application of Lemma 2.21 and Lemma 2.22 shows that the functions  $F_{q,0,A}$  and  $F^{0,y,A}$  without the oscillating factors lie in  $\underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ . But  $(p, x) \mapsto e^{-i\langle p, y \rangle}$  and  $(p, x) \mapsto e^{-i\langle q, x \rangle}$  are scalar symbols of type 0 and order 0, as is easily verified by differentiation. Hence Corollary 2.8 yields  $F_{q,y,A}, F^{q,y,A} \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ . To compare the oscillatory integrals of these functions, we pick  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, \mathbb{R})$  as in Proposition 3.4, and compute according to (3.15)

$$\begin{aligned} (2\pi)^n I(F_{q,y,A}) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} e^{-i\langle p, y \rangle} \chi(\varepsilon p, \varepsilon x) F(Ap + q, x) \\ &= \lim_{\varepsilon \rightarrow 0} |\det A|^{-1} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle A^{-1}(p-q), x \rangle} \chi(\varepsilon A^{-1}p, \varepsilon(x+y)) F(p, x+y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \chi(\varepsilon A^{-1}p, \varepsilon(A^T x + y)) e^{-i\langle q, x \rangle} F(p, A^T x + y). \end{aligned}$$

Since also  $(p, x) \mapsto \chi(A^{-1}p, A^T x)$  is a smooth, compactly supported function which is equal to 1 on an open neighborhood of the origin, we can use Proposition 3.4 again to conclude that the last line coincides with  $(2\pi)^n I(F^{q,y,A})$ .  $\square$

For the next statement, we represent the bilinear form used in the oscillating factor as  $\langle p, x \rangle = (p, Mx)$  with some  $M \in GL(n, \mathbb{R})$ ,  $|\det M| = 1$ , and the standard Euclidean inner product  $(\cdot, \cdot)$  on  $\mathbb{R}^n$ . The transpose of  $M$  with respect to this inner product will be denote  $M^T$ .

**Proposition 3.8** *Let  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$  and  $\mu \in \mathbb{N}_0^n$ . Then  $\partial_p^\mu F$ ,  $\partial_x^\mu F$ ,  $(Mx)^\mu F$ ,  $(M^T p)^\mu F$  lie in  $\underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ , and*

$$I(\partial_p^\mu F) = (-i)^{|\mu|} I((Mx)^\mu F), \quad I(\partial_x^\mu F) = (-i)^{|\mu|} I((M^T p)^\mu F). \quad (3.24)$$

Proof. For  $F \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$ , the claimed equations amount to an integration by parts against the oscillating factor  $e^{i\langle p, x \rangle} = e^{i\langle p, Mx \rangle}$ , since

$$\partial_p^\mu e^{i\langle p, Mx \rangle} = i^{|\mu|} (Mx)^\mu e^{i\langle p, Mx \rangle}, \quad \partial_x^\mu e^{i\langle p, Mx \rangle} = i^{|\mu|} (M^T p)^\mu e^{i\langle p, Mx \rangle}.$$

Thus we only have to show that the functions on the left and right hand side in (3.24) are symbols in  $\underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ , and can be approximated by compactly supported symbols. So let  $F \in \mathbf{S}^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$  for some order  $\mathbf{m}$  and type  $-1 < \boldsymbol{\rho} \leq 1$  for a defining system of seminorms  $\mathcal{Q}$  for  $V$ , and pick a sequence  $F_n \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, V)$  converging to  $F$  in the  $\mathbf{S}^{\mathbf{m}', \boldsymbol{\rho}'}$ -topology for some  $\mathbf{m}' > \mathbf{m}$ ,  $-1 < \boldsymbol{\rho}' \leq \boldsymbol{\rho}$ . Then, according to Proposition 2.4,  $\partial_p^\mu F_n \rightarrow \partial_p^\mu F$  in the  $\mathbf{S}^{\mathbf{m}' - \boldsymbol{\rho}'|\mu|, \boldsymbol{\rho}'}$ -topology. But as  $\boldsymbol{\rho}' > -1$ , we can apply Proposition 2.3 *iv.*) to see that this sequence also converges in the  $\mathbf{S}^{\mathbf{m}' + |\mu|, \boldsymbol{\rho}'}$ -topology. Hence, by definition of  $I$ , we have  $I(\partial_p^\mu F) = \lim_n I_0(\partial_p^\mu F_n)$ . Concerning the right hand side in (3.24), note that  $(Mx)^\mu$  is a scalar symbol of order  $|\mu|$  and type 1. Hence, using Corollary 2.8 and  $\boldsymbol{\rho} \leq 1$ , we see that  $(Mx)^\mu F$  is a symbol of order  $\mathbf{m} + |\mu|$  and type  $\boldsymbol{\rho}$ , and thus an element of  $\underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$ . Moreover,  $(Mx)^\mu F_n \rightarrow (Mx)^\mu F$  in the  $\mathbf{S}^{\mathbf{m}' + |\mu|, \boldsymbol{\rho}'}$ -topology. Thus  $I((Mx)^\mu F) = \lim_n I_0((Mx)^\mu F_n)$ , and the first identity in (3.24) follows. The proof of the second identity is completely analogous.  $\square$

We next compute the oscillatory integrals of symbols  $(p, x) \mapsto F(p, x)$  which are constant in either  $x$  or  $p$ . The following result also explains our choice of normalization factor  $(2\pi)^{-n}$  in the definition (3.1) of the oscillatory integral.

**Proposition 3.9** *Let  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$  satisfy  $F(p, x) = F(0, x)$  or  $F(p, x) = F(p, 0)$  for all  $p, x \in \mathbb{R}^n$ . Then*

$$I(F) = F(0). \quad (3.25)$$

*In particular, constant symbols  $v : (p, x) \mapsto v$ ,  $v \in V$ , have oscillatory integral  $I(v) = v$ .*

Proof. We present only the proof for the claims about a symbol  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, V)$  satisfying  $F(p, x) = F(0, x)$ ; the arguments for the other case are analogous. To evaluate the oscillatory integral of  $F$ , let  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$ , with  $\chi(x) = 1$  for  $|x| \leq 1$ , and let  $\tilde{\chi}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} dp e^{i\langle p, x \rangle} \chi(p)$  denote the Fourier transform of  $\chi$  with respect to the chosen inner product. Then Proposition 3.4 can be applied with the product cutoff function  $(p, x) \mapsto \chi(p)\chi(x)$ , and we obtain

$$\begin{aligned} I(F) &= \lim_{\varepsilon \rightarrow 0} (2\pi)^{-n} \int_{\mathbb{R}^n} dp \int_{\mathbb{R}^n} dx e^{i\langle p, x \rangle} \chi(\varepsilon p) \chi(\varepsilon x) F(0, x) \\ &= \lim_{\varepsilon \rightarrow 0} (2\pi)^{-n} \varepsilon^{-n} \int_{\mathbb{R}^n} dp \int_{\mathbb{R}^n} dx e^{i\langle p, x/\varepsilon \rangle} \chi(p) \chi(\varepsilon x) F(0, x) \\ &= \lim_{\varepsilon \rightarrow 0} (2\pi)^{-n/2} \varepsilon^{-n} \int_{\mathbb{R}^n} dx \tilde{\chi}(x/\varepsilon) \chi(\varepsilon x) F(0, x) \end{aligned}$$



$$= \lim_{\varepsilon \rightarrow 0} (2\pi)^{-n/2} \int_{\mathbb{R}^n} dx \tilde{\chi}(x) \chi(\varepsilon^2 x) F(0, \varepsilon x) .$$

To show that this limit coincides with  $F(0)$ , let  $\mathcal{Q}$  be a defining system of seminorms on  $V$ , with order  $\mathbf{m}$  and type  $-1 < \boldsymbol{\rho} \leq 1$ , such that  $F \in \mathbf{S}^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$ . For any  $q \in \mathcal{Q}$ , we have the estimate

$$\begin{aligned} q \left( \int_{\mathbb{R}^n} dx \tilde{\chi}(x) \chi(\varepsilon^2 x) F(0, \varepsilon x) - \int_{\mathbb{R}^n} dx \tilde{\chi}(x) F(0) \right) \\ \leq \int_{\mathbb{R}^n} dx |\tilde{\chi}(x)| q \left( \chi(\varepsilon^2 x) F(0, \varepsilon x) - F(0) \right) . \end{aligned}$$

As the Fourier transform of a smooth, compactly supported function,  $\tilde{\chi}$  is an element of  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ , and since  $F \in \mathbf{S}^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$ , and  $\chi$  is bounded, we easily find a scalar integrable function  $g$  such that  $|\tilde{\chi}(x)| q(\chi(\varepsilon^2 x) F(\varepsilon x) - F(0)) \leq g(x)$  for all  $\varepsilon \leq 1$ . Hence the limit  $\varepsilon \rightarrow 0$  of the right hand side of the above estimate can be evaluated by dominated convergence, and since  $\chi(0) = 1$ , this limit is zero. As  $q$  was arbitrary, we have in the topology of  $V$

$$I(F) = \lim_{\varepsilon \rightarrow 0} (2\pi)^{-n/2} \int_{\mathbb{R}^n} dx \tilde{\chi}(x) \chi(\varepsilon^2 x) F(0, \varepsilon x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} dx \tilde{\chi}(x) F(0) .$$

But in view of our normalization of the inner product  $\langle \cdot, \cdot \rangle$ , the inverse Fourier transform gives  $(2\pi)^{-n/2} \int dx \tilde{\chi}(x) = \chi(0) = 1$ . So we arrive at the claimed identity

$$I(F) = \chi(0) F(0) = F(0) .$$

The statement about constant symbols follows by choosing  $F$  constant.  $\square$

As the last property of oscillatory integrals needed in our applications, we discuss a Fubini type theorem for multiple oscillatory integrals. To this end, we consider symbols  $F \in \underline{\mathbf{S}}(\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}, V)$  depending on two pairs of variables  $(p_1, x_1), (p_2, x_2)$ , with  $p_j, x_j \in \mathbb{R}^{n_j}$ ,  $j = 1, 2$ . Our discussion of multiple oscillatory integrals is greatly facilitated by Proposition 2.29, stating that for  $F \in \underline{\mathbf{S}}(\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}, V)$ , the maps

$$F_1: \mathbb{R}^{2n_1} \longrightarrow \underline{\mathbf{S}}(\mathbb{R}^{2n_2}, V) \quad \text{and} \quad F_2: \mathbb{R}^{2n_2} \longrightarrow \underline{\mathbf{S}}(\mathbb{R}^{2n_1}, V), \quad (3.26)$$

given by

$$F_1(p_1, x_1): (p_2, x_2) \mapsto F(p_1, x_1; p_2, x_2) \quad \text{and} \quad F_2(p_2, x_2): (p_1, x_1) \mapsto F(p_1, x_1; p_2, x_2),$$

respectively, are symbols in their own right.

Subsequently we have to distinguish different kinds of oscillatory integrals. On  $\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}$  we use the induced pairing

$$\langle p_1 \oplus p_2, x_1 \oplus x_2 \rangle = \langle p_1, x_1 \rangle + \langle p_2, x_2 \rangle . \quad (3.27)$$

Hence the oscillatory integrals over symbols  $F \in \underline{\mathbf{S}}(\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}, V)$  will be carried out with respect to the oscillating factor  $e^{i\langle p_1, x_1 \rangle} e^{i\langle p_2, x_2 \rangle}$ , and denoted by  $I$  as before. On the other hand, oscillatory integrals over the symbols (3.26) which are defined on  $\mathbb{R}^{2n_1}$  (respectively  $\mathbb{R}^{2n_2}$ ), and take values in some other symbol space  $\mathbf{S}^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n_2}, V)$  (respectively  $\mathbf{S}^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n_1}, V)$ ), will be carried out with respect to the oscillating factors  $e^{i\langle p_1, x_1 \rangle}$  (respectively  $e^{i\langle p_2, x_2 \rangle}$ ), and denoted  $\hat{I}_1$  (respectively  $\hat{I}_2$ ).



**Proposition 3.10** *Let  $V$  be a sequentially complete locally convex space. For any  $F \in \underline{\mathbb{S}}(\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}, V)$ , we have  $\hat{I}_1(F_1) \in \underline{\mathbb{S}}(\mathbb{R}^{2n_2}, V)$ ,  $\hat{I}_2(F_2) \in \underline{\mathbb{S}}(\mathbb{R}^{2n_1}, V)$ , with*

$$I_2(\hat{I}_1(F_1)) = I_1(\hat{I}_2(F_2)) = I(F). \quad (3.28)$$

*Equivalently,*

$$\begin{aligned} & (2\pi)^{-n_2} \int_{\mathbb{R}^{2n_2}} dp_2 dx_2 e^{i\langle p_2, x_2 \rangle} \left( (2\pi)^{-n_1} \int_{\mathbb{R}^{2n_1}} dp_1 dx_1 e^{i\langle p_1, x_1 \rangle} F(p_1, x_1; p_2, x_2) \right) \\ &= (2\pi)^{-n_1} \int_{\mathbb{R}^{2n_1}} dp_1 dx_1 e^{i\langle p_1, x_1 \rangle} \left( (2\pi)^{-n_2} \int_{\mathbb{R}^{2n_2}} dp_2 dx_2 e^{i\langle p_2, x_2 \rangle} F(p_1, x_1; p_2, x_2) \right) \\ &= (2\pi)^{-(n_1+n_2)} \int_{\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}} dp dx e^{i\langle p, x \rangle} F(p, x) \end{aligned}$$

Proof. Let  $\mathcal{Q}$  be a defining system of seminorms for  $V$ , and  $\mathbf{m}$ ,  $-1 < \boldsymbol{\rho} \leq 1$  an order and a type for  $\mathcal{Q}$  such that  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}, V)$ . According to Proposition 2.29,  $F_1 \in S^{\hat{\mathbf{m}}, \hat{\boldsymbol{\rho}}}(\mathbb{R}^{2n_1}, S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n_2}, V))$ ,  $F_2 \in S^{\hat{\mathbf{m}}, \hat{\boldsymbol{\rho}}}(\mathbb{R}^{2n_2}, S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n_1}, V))$  with the order  $\hat{\mathbf{m}}$  and type  $\hat{\boldsymbol{\rho}}$  defined in (2.63). Note that since  $-1 < \hat{\boldsymbol{\rho}} \leq 1$ , the oscillatory integrals  $\hat{I}_1(F_1) \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n_2}, V) \subset \underline{\mathbb{S}}(\mathbb{R}^{2n_2}, V)$  and  $\hat{I}_2(F_2) \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^{2n_1}, V) \subset \underline{\mathbb{S}}(\mathbb{R}^{2n_1}, V)$  exist. Hence all integrals in (3.28) are well-defined. To show that they coincide, we can argue with the usual continuity and approximation techniques: for compactly supported symbols the integrals coincide by the Fubini theorem for Riemann integrals. Then the continuity statements of Proposition 2.29 and Theorem 3.2 give the equality for all symbols.  $\square$

## 4 Rieffel deformations for polynomially bounded $\mathbb{R}^n$ -actions

We now apply the symbol calculus developed so far to extend Rieffel's deformation of Fréchet algebras with isometric  $\mathbb{R}^n$ -actions [23] to a more general setting. As before, we will consider functions taking values in locally convex sequentially complete vector spaces  $V$ . We will in this chapter always assume a filtrating defining system  $\mathcal{Q}$  of seminorms for  $V$ . The symbols we are interested in will be generated with the help of suitable  $\mathbb{R}^n$ -actions, and we introduce some standard notation first.

For an  $\mathbb{R}^n$ -action  $\alpha: \mathbb{R}^n \times V \longrightarrow V$ , we consider the functions

$$\alpha(v): \mathbb{R}^n \longrightarrow V, \quad x \mapsto \alpha_x(v) \quad (4.1)$$

for  $v \in V$ . The action will be called strongly smooth if  $\alpha(v) \in \mathcal{C}^\infty(\mathbb{R}^n, V)$  for all  $v \in V$ . Its derivatives at  $x = 0$  are denoted by

$$X^\mu: V \longrightarrow V, \quad X^\mu v := \partial_x^\mu \alpha_x(v)|_{x=0}, \quad (4.2)$$

where  $\mu \in \mathbb{N}_0^n$  as usual. All actions will be assumed to act by linear maps  $\alpha_x: V \longrightarrow V$ . If  $\alpha$  is strongly smooth and the  $\alpha_x$  are continuous for all  $x \in \mathbb{R}^n$ , then one has  $\partial_x^\mu \alpha_x(v) = X^\mu \alpha_x v = \alpha_x X^\mu v$ .

**Definition 4.1** Let  $V$  be a sequentially complete locally convex space with defining system of seminorms  $\mathcal{Q}$ , and let  $\mathbf{m}$  be an order for  $\mathcal{Q}$ . A smooth polynomially bounded  $\mathbb{R}^n$ -action (of order  $\mathbf{m}$ ) is an action  $\alpha: \mathbb{R}^n \times V \longrightarrow V$  such that

- i.)  $\alpha(v) \in S^{\mathbf{m},0}(\mathbb{R}^n, V)$  for each  $v \in V$ .
- ii.)  $V \ni v \mapsto \alpha(v) \in S^{\mathbf{m},0}(\mathbb{R}^n, V)$  is continuous, i.e. for any  $q \in \mathcal{Q}$ ,  $\mu \in \mathbb{N}_0^n$ , there exists  $q' \in \mathcal{Q}$ , such that

$$\|\alpha(v)\|_{q,\mu}^{\mathbf{m},0} \leq q'(v) \quad (4.3)$$

for all  $v \in V$ .

Smooth polynomially bounded actions can equivalently be characterized as follows.

**Lemma 4.2** Let  $\alpha$  be a strongly smooth action on  $V$ . Then  $\alpha$  is polynomially bounded of order  $\mathbf{m}$  in the sense of Definition 4.1 if and only if the following two conditions are satisfied:

- i.) For each  $q \in \mathcal{Q}$ , there exists  $q' \in \mathcal{Q}$  such that

$$q(\alpha_x(v)) \leq (1 + \|x\|^2)^{\frac{1}{2}\mathbf{m}(q)} q'(v) \quad (4.4)$$

for all  $x \in \mathbb{R}^n$ ,  $v \in V$ .

- ii.) The derivatives  $X^\mu: V \longrightarrow V$  are continuous.

If  $V$  is a Fréchet space and  $\mathcal{Q}$  is chosen countable then  $\alpha$  is polynomially bounded of order  $\mathbf{m}$  if and only if just the first condition is satisfied.

Proof. Let  $\alpha$  be a smooth polynomially bounded action of order  $\mathbf{m}$ . Then its defining properties imply that for any  $q \in \mathcal{Q}$  there exists  $q' \in \mathcal{Q}$  such that for all  $x \in \mathbb{R}^n$ ,  $v \in V$ ,

$$q(\alpha_x(v)) \leq (1 + \|x\|^2)^{\frac{1}{2}\mathbf{m}(q)} \|\alpha(v)\|_{q,0}^{\mathbf{m},0} \leq (1 + \|x\|^2)^{\frac{1}{2}\mathbf{m}(q)} q'(v),$$

i.e. (4.4) holds. Furthermore, given  $q \in \mathcal{Q}$ ,  $\mu \in \mathbb{N}_0^n$ , there exists  $q' \in \mathcal{Q}$  such that

$$q(X^\mu v) = q(\partial_x^\mu \alpha_x(v)|_{x=0}) \leq \sup_{x \in \mathbb{R}^n} \frac{q(\partial_x^\mu \alpha_x(v))}{(1 + \|x\|^2)^{\frac{1}{2}\mathbf{m}(q)}} = \|\alpha(v)\|_{q,\mu}^{\mathbf{m},0} \leq q'(v)$$

for all  $v \in V$ . Hence  $X^\mu: V \longrightarrow V$  is continuous. Now assume that  $\alpha$  is a strongly smooth action satisfying the two conditions listed in this lemma. Then to any  $q \in \mathcal{Q}$ ,  $\mu \in \mathbb{N}_0^n$ , there exist  $q', q'' \in \mathcal{Q}$  such that

$$q(\partial_x^\mu \alpha_x(v)) = q(\alpha_x(X^\mu v)) \leq (1 + \|x\|^2)^{\frac{1}{2}\mathbf{m}(q)} q'(X^\mu v) \leq (1 + \|x\|^2)^{\frac{1}{2}\mathbf{m}(q)} q''(v)$$

for all  $x \in \mathbb{R}^n$ ,  $v \in V$ . This shows both,  $\alpha(v) \in S^{\mathbf{m},0}(V)$ , and the continuity of  $v \mapsto \alpha(v)$ . Hence Definition 4.1 and the two conditions in this lemma are equivalent. We now consider the special but important case that  $V$  is a Fréchet space. Thus assume that  $\mathcal{Q}$  is countable. When equipped with the family of seminorms  $\mathcal{Q}^\infty := \{q_\mu := q \circ X^\mu \mid q \in \mathcal{Q}, \mu \in \mathbb{N}_0^n\}$ , this space will be called  $V^\infty$ . Also  $V^\infty$  is a Fréchet space. As linear spaces,  $V = V^\infty$ , and clearly, the identity  $\text{id}: V^\infty \longrightarrow V$  is linear, continuous

and bijective. Hence we can apply the open mapping theorem for Fréchet spaces to conclude that  $\text{id}: V \rightarrow V^\infty$  is continuous as well, i.e.  $V = V^\infty$  as Fréchet spaces. But this is equivalent to the derivatives  $X^\mu: V \rightarrow V$  being continuous, i.e. condition *ii.*) is automatically satisfied.  $\square$

**Remark 4.3** *i.)* A polynomially bounded action  $\alpha$  acts by continuous maps  $\alpha_x$ , as can be read off from (4.4).

*ii.)* The arguments in the above lemma also explain why we consider only actions of type  $\rho = 0$  here. For if  $\alpha(v) \in S^{\mathbf{m}, \rho}(V)$  and the derivatives  $X^\mu$  are continuous, we can argue as above to show that  $\alpha(v)$  is actually of type 0.

In Rieffel's original approach,  $V$  is taken to be a Fréchet algebra with a strongly continuous  $\mathbb{R}^n$ -action  $\alpha$  by automorphisms  $\alpha_x$  which are isometric for all  $x \in \mathbb{Q}$ . On the subspace  $V^\infty$  of all smooth vectors for  $\alpha$ , the  $\alpha(v)$  are then symbols of order 0 and type 0, see [23]. Using the symbol calculus of the preceding sections, we will now extend many of the results of Rieffel to the case where  $\alpha(v) \in S^{\mathbf{m}, 0}(\mathbb{R}^n, V)$  for an arbitrary order  $\mathbf{m}$ . In Section 5, we will then provide various examples of smooth polynomially bounded  $\mathbb{R}^n$ -actions.

In comparison to [23], we will take here also a somewhat more general point of view concerning the algebraic structure, which involves three sequentially complete locally convex spaces  $V, W, U$  with filtrating defining systems of seminorms  $Q^V, Q^W, Q^U$ . Each of these spaces is equipped with a smooth polynomially bounded  $\mathbb{R}^n$ -action  $\alpha^V, \alpha^W, \alpha^U$  of order  $\mathbf{m}^V, \mathbf{m}^W, \mathbf{m}^U$ , respectively, and the derivatives with respect to these actions will be denoted  $X_U^\mu, X_V^\mu, X_W^\mu$ .

In this setting, we consider a bilinear map

$$\mu: V \times W \rightarrow U \quad (4.5)$$

which is required to be covariant in the sense that

$$\alpha_x^U \mu(v, w) = \mu(\alpha_x^V v, \alpha_x^W w), \quad v \in V, w \in W, x \in \mathbb{R}^n. \quad (4.6)$$

In many applications  $\mu$  will be jointly continuous, but in some cases we also need to work with a bilinear map  $\mu$  which is only separately continuous. In the following we will therefore always only assume that  $\mu$  is separately continuous, and explicitly point out when we consider the special case that  $\mu$  is jointly continuous.

This setting includes the case where  $\mathcal{A} := V = W = U$  is an algebra with (separately) continuous product  $\mu$ , and  $\alpha := \alpha^V = \alpha^W = \alpha^U$  acts by automorphisms. But the more general formulation allows, for example, to also consider covariant modules, where  $\mathcal{A} := V$  is taken to be an algebra and  $\mathcal{E} := W = U$  is a left  $\mathcal{A}$ -module with a smooth  $\mathbb{R}^n$ -action  $\beta := \alpha^W = \alpha^U$ , and (separately) continuous module structure  $\mu: \mathcal{A} \times \mathcal{E} \rightarrow \mathcal{E}$  satisfying (4.6). This setup will therefore be suitable for the discussion of deformations of algebras and their covariant modules. In the following, we will always assume without further mentioning that spaces  $V, W, U$ , actions  $\alpha^V, \alpha^W, \alpha^U$ , and a bilinear map  $\mu$  with the specified properties are given.

Following Rieffel, we now consider a real  $(n \times n)$ -matrix  $\theta$  as our deformation parameter, and introduce the functions,  $v \in V$ ,  $w \in W$ ,

$$\mu_{vw}^\theta: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow U, \quad \mu_{vw}^\theta(p, x) := \mu(\alpha_{\theta p}^V v, \alpha_x^W w). \quad (4.7)$$

As  $\alpha^V$ ,  $\alpha^W$  are smooth and polynomially bounded, it follows from Proposition 2.5, *ii.*), that these functions are symbols in  $\underline{S}(\mathbb{R}^{2n}, U)$  if  $\mu$  is jointly continuous. If  $\mu$  is however only separately continuous, more work is needed to arrive at this conclusion. We begin with the following lemma.

**Lemma 4.4** *Let  $\alpha$  be a smooth polynomially bounded action on  $V$ . Moreover, let  $F \in \mathcal{C}^\infty(\mathbb{R}^n, V)$ . Then*

$$F^\alpha: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow V, \quad F^\alpha(p, x) := \alpha_p(F(x))$$

*is smooth.*

Proof. First we show that  $F^\alpha$  is continuous. To this end, we make use of the differentiability of  $\alpha$ , which implies that for any  $q \in \mathcal{Q}$ , there exists  $q' \in \mathcal{Q}$  and a continuous function  $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$  such that for all  $p, p' \in \mathbb{R}^n$ ,  $v \in V$

$$q(\alpha_p(v) - \alpha_{p'}(v)) \leq \|p - p'\| f(p, p') q'(v).$$

The proof of this estimate can be carried out along the same lines as in Lemma 2.25. With this bound we find,  $p, p', x, x' \in \mathbb{R}^n$ ,

$$\begin{aligned} q(F^\alpha(p, x) - F^\alpha(p', x')) &\leq q(\alpha_p(F(x)) - \alpha_{p'}(F(x))) + q(\alpha_{p'}(F(x)) - \alpha_{p'}(F(x'))) \\ &\leq \|p - p'\| f(p, p') q'(F(x)) + (1 + \|p'\|^2)^{\frac{1}{2}m(q)} q''(F(x) - F(x')), \end{aligned}$$

with some  $q', q'' \in \mathcal{Q}$ . The continuity of  $F^\alpha$  is then clear. Furthermore,  $(p, x) \mapsto F^\alpha(p, x)$  is separately smooth in  $p$  and  $x$  (in  $x$  because the  $\alpha_p$  are linear and continuous), with partial derivatives

$$\partial_p^\nu F^\alpha(p, x) = \alpha_p X^\nu F(x), \quad \partial_x^\nu F^\alpha(p, x) = \alpha_p \partial_x^\nu F(x).$$

According to Lemma 4.2, the derivatives  $X^\nu: V \longrightarrow V$  are continuous. Thus  $x \mapsto X^\nu F(x)$  is smooth, and clearly,  $x \mapsto \partial_x^\nu F(x)$  is smooth as well. Hence the partial derivatives of  $F^\alpha$  are of the same form as  $F^\alpha$ , and thus in particular continuous. This implies that  $F^\alpha$  is smooth.  $\square$

**Lemma 4.5** *Let  $v \in V$ ,  $w \in W$  and consider  $F_{vw}, G_{vw}: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow U$  defined by*

$$F_{vw}(p, x) := \alpha_p^U(\mu(v, \alpha_x^W(w))) \quad \text{and} \quad G_{vw}(p, x) := \alpha_x^U(\mu(\alpha_p^V(v), w)). \quad (4.8)$$

*Then there exists an order  $\hat{m}$  on  $\mathcal{Q}^U$  such that  $F_{vw}, G_{vw} \in S^{\hat{m}, 0}(\mathbb{R}^{2n}, U)$ , and for fixed  $v_0 \in V$ ,  $w_0 \in W$ , the mappings*

$$W \ni w \mapsto F_{v_0 w} \in S^{\hat{m}, 0}(\mathbb{R}^{2n}, U)$$

$$V \ni v \mapsto G_{vw_0} \in S^{\hat{\mathbf{m}},0}(\mathbb{R}^{2n}, U)$$

are linear and continuous.

Proof. We will only prove the statements about  $F_{vw}$  as the discussion of  $G_{vw}$  is completely analogous. In view of the separate continuity of  $\mu$ , the map  $w \mapsto \mu(v, w)$  is continuous for fixed  $v$ , and as  $\alpha^W$  is a smooth action, we see that  $F_{vw}$  is of the form considered in the previous lemma and hence smooth. Its partial derivatives are,  $\nu, \kappa \in \mathbb{N}_0^n$ ,

$$\partial_p^\nu \partial_x^\kappa F_{vw}(p, x) = \alpha_p^U \left( X_U^\nu \mu(v, \alpha_x^W(X_W^\kappa w)) \right).$$

To estimate these derivatives, let  $\mathbf{q} \in \mathcal{Q}^U$ . Then there exists  $\mathbf{q}' \in \mathcal{Q}^U$  such that

$$\mathbf{q}(\partial_p^\nu \partial_x^\kappa F_{vw}(p, x)) \leq (1 + \|p\|^2)^{\frac{1}{2}\mathbf{m}^U(\mathbf{q})} \mathbf{q}'(X_U^\nu \mu(v, \alpha_x^W X_W^\kappa w)).$$

As  $w \mapsto \mu(v, w)$  is continuous and  $U = U^\infty$ ,  $W = W^\infty$  as locally convex spaces, we now find  $\mathbf{q}'', \mathbf{q}''', \mathbf{q}'''' \in \mathcal{Q}^W$ ,  $\sigma \in \mathbb{N}_0^n$ , and constants  $c, C_v > 0$  such that

$$\begin{aligned} \mathbf{q}(\partial_p^\nu \partial_x^\kappa F_{vw}(p, x)) &\leq (1 + \|p\|^2)^{\frac{1}{2}\mathbf{m}^U(\mathbf{q})} C_v \mathbf{q}''(X_W^\sigma X_W^\kappa \alpha_x^W w) \\ &\leq C_v (1 + \|p\|^2)^{\frac{1}{2}\mathbf{m}^U(\mathbf{q})} (1 + \|x\|^2)^{\frac{1}{2}\mathbf{m}^W(\mathbf{q}'')} \mathbf{q}'''(X_W^\sigma X_W^\kappa w) \\ &\leq C_v c (1 + \|p\|^2 + \|x\|^2)^{\frac{1}{2}(|\mathbf{m}^U(\mathbf{q})| + |\mathbf{m}^W(\mathbf{q}'')|)} \mathbf{q}''''(w) \end{aligned}$$

for all  $p, x \in \mathbb{R}^n$ ,  $w \in W$ . Here  $c$  depends on  $\kappa, \nu, \mathbf{q}$ , but not on  $v, w, p, x$ . This estimate shows that  $F_{vw}$  is a symbol in  $S^{\hat{\mathbf{m}},0}(\mathbb{R}^{2n}, U)$ , with  $\hat{\mathbf{m}}(\mathbf{q}) := |\mathbf{m}^U(\mathbf{q})| + |\mathbf{m}^W(\mathbf{q}'')|$  (cf. Proposition 2.5). Moreover,

$$\|F_{vw}\|_{\mathbf{q}, \nu \oplus \kappa}^{\hat{\mathbf{m}},0} \leq C_v c \mathbf{q}''''(w),$$

that is,  $w \mapsto F_{vw}$  is continuous for fixed  $v$ .  $\square$

After these preparations we can derive the following basic statement about the deformation of  $\mu$ .

**Proposition 4.6** *Let  $v \in V$ ,  $w \in W$ , and  $\theta \in \mathbb{R}^{n \times n}$ .*

- i.) *The functions  $\mu_{vw}^\theta$  are symbols in  $\underline{S}(\mathbb{R}^{2n}, U)$ .*
- ii.) *The maps defined by their oscillatory integrals*

$$V \times W \ni (v, w) \mapsto \mu_\theta(v, w) := I_U(\mu_{vw}^\theta) \in U \quad (4.9)$$

*are bilinear and (separately) continuous if  $\mu$  is (separately) continuous.*

- iii.)  *$\mu_\theta$  satisfies the covariance property (4.6).*

Proof. Let  $v \in V$ ,  $w \in W$  and  $\theta$  be fixed. Thanks to the covariance of  $\mu$  (4.6), we have

$$\mu_{vw}^\theta(p, x) = F_{vw}(\theta p, x - \theta p) = G_{vw}(x, \theta p - x).$$

As  $F_{vw}, G_{vw} \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, U)$ , an application of Lemma 3.7 shows  $\mu_{vw}^\theta \in \underline{\mathbb{S}}(\mathbb{R}^{2n}, U)$  and hence the first part. By the preceding lemma, we see that

$$V \times W \ni (v, w) \mapsto \mu_{vw}^\theta \in S^{\hat{\mathbf{m}}, 0}(\mathbb{R}^{2n}, U)$$

is separately continuous for some order  $\hat{\mathbf{m}}$  on  $\mathcal{Q}^U$ . Hence the oscillatory integral  $I_U(\mu_{vw}^\theta)$  exists and depends separately continuous on  $v, w$ . The bilinearity of  $\mu^\theta$  is clear. In case  $\mu$  is jointly continuous, note that by assumption,  $\alpha_\theta^V(v) \in S^{\mathbf{m}^V, 0}(\mathbb{R}^n, V)$  and  $\alpha^W(w) \in S^{\mathbf{m}^W, 0}(\mathbb{R}^n, W)$  with orders  $\mathbf{m}^V, \mathbf{m}^W$  for  $\mathcal{Q}^V, \mathcal{Q}^W$ . The maps  $v \mapsto \alpha_\theta^V(v)$  and  $w \mapsto \alpha^W(w)$  are continuous from  $V$  (respectively  $W$ ) to  $S^{\mathbf{m}^V, 0}(\mathbb{R}^n, V)$  (respectively  $S^{\mathbf{m}^W, 0}(\mathbb{R}^n, W)$ ) by Definition 4.1, *ii.*). Furthermore, according to Proposition 2.5, *ii.*,  $\alpha_\theta^V(v), \alpha^W(w) \mapsto \mu(\alpha_\theta^V(v), \alpha^W(w))$  maps  $S^{\mathbf{m}^V, 0}(\mathbb{R}^n, V) \times S^{\mathbf{m}^W, 0}(\mathbb{R}^n, W)$  continuously into  $S^{\mathbf{m}', 0}(\mathbb{R}^{2n}, U)$ , with some order  $\mathbf{m}'$  for  $\mathcal{Q}^U$ . Finally, the oscillatory integral maps  $\mu_{vw}^\theta \mapsto \mu_\theta(v, w)$  continuously from  $S^{\mathbf{m}', 0}(\mathbb{R}^{2n}, U)$  to  $U$  by Theorem 3.2. As a composition of these continuous maps,  $\mu_\theta: V \times W \rightarrow U$  is therefore continuous, too. This completes the second part. To check the covariance property (4.6), note that since  $\alpha_x^U$  is continuous for each  $x \in \mathbb{R}^n$  by Remark 4.3 *i.*), it can be pulled inside the oscillatory integral defining  $\mu_\theta$  according to Lemma 3.5. Since (4.6) holds for  $\mu$ , and  $\alpha^V, \alpha^W$  are  $\mathbb{R}^n$ -actions, it follows that  $\mu_\theta$  satisfies (4.6) as well:

$$\begin{aligned} \alpha_x^U \mu_\theta(v, w) &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dy e^{i\langle p, y \rangle} \alpha_x^U \mu(\alpha_{\theta q}^V(v), \alpha_y^W(w)) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dy e^{i\langle p, y \rangle} \mu(\alpha_{\theta q+x}^V(v), \alpha_{y+x}^W(w)) \\ &= \mu_\theta(\alpha_x^V(v), \alpha_x^W(w)). \end{aligned}$$

□

Depending on the context,  $\mu_\theta$  from (4.9) will be referred to as the *deformed product*, *deformed module structure*, or just *deformed bilinear map*.

In the next proposition we justify these names by demonstrating the most basic feature of a deformation, namely that it reduces to the identity for vanishing deformation parameter.

**Proposition 4.7** *Let  $\theta, \theta' \in \mathbb{R}^{n \times n}$ .*

*i.) For  $\theta = 0$ , we have  $\mu_0 = \mu$ .*

*ii.)  $(\mu_\theta)_{\theta'} = \mu_{\theta+\theta'}$ .*

Proof. Let  $v \in V, w \in W$ . For  $\theta = 0$ , the symbol  $(p, x) \mapsto \mu(\alpha_{\theta p}^V v, \alpha_x^W w)$  (4.7) is independent of  $p$ . Hence Proposition 3.9 applies, and we have  $\mu_0(v, w) = \mu_\theta(\alpha_0^V(v), \alpha_0^W(w)) = \mu(v, w)$ , showing the first part. For the second part, by Proposition 4.6,  $\mu_\theta$  has the same properties as  $\mu$ , so  $(\mu_\theta)_{\theta'}$  is well-defined. Using successively the definition of  $\mu_\theta$ , the substitution  $x \rightarrow x - x'$  according to Lemma 3.7 and Fubini's theorem in form of Proposition 3.10, we compute

$$(\mu_\theta)_{\theta'}(v, w)$$

$$\begin{aligned}
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \mu_\theta(\alpha_{\theta'p'}^V(v), \alpha_{x'}^W(w)) \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \left( (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \mu(\alpha_{\theta p + \theta' p'}^V(v), \alpha_{x+x'}^W(w)) \right) \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \left( (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} e^{-i\langle p, x' \rangle} \mu(\alpha_{\theta p + \theta' p'}^V(v), \alpha_x^W(w)) \right) \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \left( (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} e^{-i\langle p, x' \rangle} \mu(\alpha_{\theta' p' + \theta p}^V(v), \alpha_x^W(w)) \right).
\end{aligned}$$

Lemma 3.7 and Proposition 3.9 show that the inner oscillatory integral has the value  $\mu(\alpha_{(\theta+\theta')p}^V(v), \alpha_x^W(w))$ . Plugging this result into the above computation gives the desired answer  $(\mu_\theta)_{\theta'}(v, w) = \mu_{\theta+\theta'}(v, w)$  by definition of  $\mu_\theta$ .  $\square$

The following lemma shows two further invariance properties of the deformation which are helpful in many situations.

**Lemma 4.8** *i.) Let  $v \in V$  and  $w \in W$ . If either  $v$  is  $\alpha^V$ -invariant or  $w$  is  $\alpha^W$ -invariant, then  $\mu_\theta(v, w) = \mu(v, w)$ .*  
*ii.) Let  $Y$  be another sequentially complete locally convex vector space, and  $T: U \rightarrow Y$  linear and continuous. If  $\theta \in \mathbb{R}^{n \times n}$  is skew-symmetric and  $T$  is  $\alpha^U$ -invariant, i.e.  $T \circ \alpha_x^U = T$  for all  $x \in \mathbb{R}^n$ , then*

$$T\mu_\theta(v, w) = T\mu(v, w). \quad (4.10)$$

Proof. For part *i.*, note that under the specified circumstances, the symbol  $(p, x) \mapsto \mu(\alpha_{\theta p}^V(v), \alpha_x^W(w))$  depends only on one of its two variables  $p, x$ . Hence Proposition 3.9 applies, and we have  $\mu_\theta(v, w) = \mu(\alpha_0^V(v), \alpha_0^W(w)) = \mu(v, w)$ . For part *ii.*, let  $v \in V$ ,  $w \in W$ . Using the continuity and linearity of  $T$  as in Lemma 3.5, as well as the covariance (4.6) and the invariance of  $T$  gives

$$\begin{aligned}
(2\pi)^n T\mu_\theta(v, w) &= \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} T\mu(\alpha_{\theta p}^V(v), \alpha_x^W(w)) \\
&= \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} T\mu(\alpha_{\theta p - x}^V(v), w).
\end{aligned}$$

Now we use Lemma 3.7 to carry out the substitution  $x \rightarrow x + \theta p$ . As  $\theta$  is skew-symmetric,  $\langle p, \theta p \rangle = 0$ , and we get

$$T\mu_\theta(v, w) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} T\mu(\alpha_{\theta p}^V(v), w).$$

This is again an oscillatory integral over a symbol which is constant in one variable, and by Proposition 3.9, we arrive at  $T\mu_\theta(v, w) = T\mu(\alpha_0^V(v), w) = T\mu(v, w)$ .  $\square$

We now consider the deformation of algebras and modules. Let  $\mathcal{A} := V$  be an algebra with separately continuous product  $\mu: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , and assume that the



(smooth, polynomially bounded)  $\mathbb{R}^n$ -action  $\alpha$  acts by automorphisms. Furthermore, let  $\mathcal{E} := W = U$  be a left  $\mathcal{A}$ -module with separately continuous module map  $\tilde{\mu}: \mathcal{A} \times \mathcal{E} \rightarrow \mathcal{E}$  and a (smooth, polynomially bounded)  $\mathbb{R}^n$ -action  $\beta$  satisfying (4.6) with  $V = \mathcal{A}$ ,  $W = U = \mathcal{E}$ ,  $\alpha^V = \alpha$ , and  $\alpha^W = \alpha^U = \beta$ . In this situation, we can deform the product  $\mu$  according to

$$\mu_\theta(a, b) := a \times_\theta b := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \tilde{\mu}(\alpha_{\theta p}(a), \alpha_x(b)), \quad a, b \in \mathcal{A}, \quad (4.11)$$

and the module structure  $\tilde{\mu}$  according to

$$\tilde{\mu}_\theta(a, \psi) := a_\theta \psi := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \tilde{\mu}(\alpha_{\theta p}(a), \beta_x(\psi)), \quad a \in \mathcal{A}, \psi \in \mathcal{E}, \quad (4.12)$$

with the same deformation parameter  $\theta \in \mathbb{R}^{n \times n}$ . We will write  $\mathcal{A}_\theta$  for the algebra given by the linear space  $\mathcal{A}$  and the product  $\times_\theta$ .

**Theorem 4.9** *Let  $\mathcal{A}$  be a sequentially complete locally convex algebra with separately continuous product  $\mu$ , and let  $\mathcal{E}$  be a sequentially complete locally convex left  $\mathcal{A}$ -module with separately continuous module structure  $\tilde{\mu}$ . Let  $\alpha$  be a smooth polynomially bounded  $\mathbb{R}^n$ -action by automorphisms on  $\mathcal{A}$ , and  $\beta$  a smooth polynomially bounded  $\mathbb{R}^n$ -action on  $\mathcal{E}$  such that*

$$\beta_x(\tilde{\mu}(a, \psi)) = \tilde{\mu}(\alpha_x(a), \beta_x(\psi)), \quad a \in \mathcal{A}, \psi \in \mathcal{E}, x \in \mathbb{R}^n. \quad (4.13)$$

i.) *In this case  $(\mathcal{E}, \tilde{\mu}_\theta)$  is a left  $\mathcal{A}_\theta$ -module, i.e.*

$$(a \times_\theta b)_\theta \psi = a_\theta b_\theta \psi, \quad a, b \in \mathcal{A}, \psi \in \mathcal{E}. \quad (4.14)$$

ii.) *If the product  $\mu$  in  $\mathcal{A}$  is associative, then so is the deformed product  $\mu_\theta$  (4.11).*

Proof. Let  $a, b \in \mathcal{A}$ ,  $\psi \in \mathcal{E}$ , and  $\theta \in \mathbb{R}^{n \times n}$ . By applying repeatedly the arguments from Lemma 4.5, one sees that

$$\mathbb{R}^{4n} \ni (p, x, p', x') \mapsto \tilde{\mu}(\alpha_{\theta p' + \theta p}(a), \tilde{\mu}(\alpha_{\theta p + x'}(b), \beta_x(\psi)))$$

is a symbol in  $\underline{\mathbb{S}}(\mathbb{R}^{4n}, \mathcal{E})$ . Its oscillatory integral can be written with the help of Fubini's theorem, the module property  $\tilde{\mu}(\mu(a, b), \psi) = \tilde{\mu}(a, \tilde{\mu}(b, \psi))$ , the separate continuity of  $\tilde{\mu}$ , and (4.6) as

$$\begin{aligned} & (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} dp dx dp' dx' e^{i\langle p, x \rangle + i\langle p', x' \rangle} \tilde{\mu}(\alpha_{\theta p' + \theta p}(a), \tilde{\mu}(\alpha_{\theta p + x'}(b), \beta_x(\psi))) \\ &= (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \left( \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \tilde{\mu}(\alpha_{\theta p' + \theta p}(a), \tilde{\mu}(\alpha_{\theta p + x'}(b), \beta_x(\psi))) \right) \\ &= (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \left( \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \tilde{\mu}(\mu(\alpha_{\theta p' + \theta p}(a), \alpha_{\theta p + x'}(b)), \beta_x(\psi)) \right) \\ &= (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \tilde{\mu} \left( \alpha_{\theta p} \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \mu(\alpha_{\theta p'}(a), \alpha_{x'}(b)), \beta_x(\psi) \right) \end{aligned}$$



$$\begin{aligned}
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \tilde{\mu}(\alpha_{\theta p}(\mu_\theta(a, b)), \beta_x(\psi)) \\
&= \tilde{\mu}_\theta(\mu_\theta(a, b), \psi).
\end{aligned}$$

On the other hand, we can use Lemma 3.7 to carry out the substitutions  $p' \rightarrow p' - p$  and  $x \rightarrow x + x'$  in the first oscillatory integral. This gives

$$\begin{aligned}
&\tilde{\mu}_\theta(\mu_\theta(a, b), \psi) \\
&= (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} dp dx dp' dx' e^{i\langle p, x \rangle + i\langle p', x' \rangle} \tilde{\mu}(\alpha_{\theta p' + \theta p}(a), \tilde{\mu}(\alpha_{\theta p + x'}(b), \beta_x(\psi))) \\
&= (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} dp dx dp' dx' e^{i\langle p, x + x' \rangle + i\langle p' - p, x' \rangle} \tilde{\mu}(\alpha_{\theta p'}(a), \tilde{\mu}(\alpha_{\theta p + x'}(b), \beta_{x+x'}(\psi))) .
\end{aligned}$$

Notice that the exponential appearing here equals  $e^{i\langle p, x \rangle + i\langle p', x' \rangle}$  because the term  $\langle p, x' \rangle$  drops out. So we can again use the covariance and separate continuity of  $\tilde{\mu}$ , and split the double oscillatory integral into two single oscillatory integrals, to arrive at the desired result,

$$\begin{aligned}
&\tilde{\mu}_\theta(\mu_\theta(a, b), \psi) \\
&= (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} dp dx dp' dx' e^{i\langle p, x \rangle + i\langle p', x' \rangle} \tilde{\mu}(\alpha_{\theta p'}(a), \beta_{x'}(\tilde{\mu}(\alpha_{\theta p}(b), \beta_x(\psi)))) \\
&= (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \tilde{\mu}\left(\alpha_{\theta p'}(a), \beta_{x'}\left(\int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \tilde{\mu}(\alpha_{\theta p}(b), \beta_x(\psi))\right)\right) \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp' dx' e^{i\langle p', x' \rangle} \tilde{\mu}(\alpha_{\theta p'}(a), \beta_{x'}(\tilde{\mu}_\theta(b, \psi))) \\
&= \tilde{\mu}_\theta(a, \tilde{\mu}_\theta(b, \psi)).
\end{aligned}$$

Rewriting  $\mu_\theta$  and  $\tilde{\mu}_\theta$  according to (4.11) and (4.12) yields (4.14). The second part follows by considering the special case  $\mathcal{E} = \mathcal{A}$ ,  $\tilde{\mu} = \mu$ ,  $\beta = \alpha$ .  $\square$

For isometric actions on Fréchet algebras, the associativity of the deformed product is known from Rieffel's work [23]. The deformation of the module structure can also be viewed as an alternative deformation of an algebra  $\mathcal{A}$  represented on  $\mathcal{E}$ , which changes the elements  $a \in \mathcal{A}$  according to  $a \mapsto a_\theta$ , but keeps the product unchanged. This deformation has been introduced under the name of *warped convolution* in the context of  $C^*$ -algebras [8, 9], it is equivalent to the deformation of the product according to (4.14).

Sticking to the setting of an algebra  $\mathcal{A}$  with product  $\mu$  and a left  $\mathcal{A}$ -module with module structure  $\tilde{\mu}$ , and actions  $\alpha, \beta$  satisfying the assumptions of Theorem 4.9, we next show how identities and star involutions behave under the deformation.

**Proposition 4.10** *Let  $\mathcal{A}$  be a locally convex sequentially complete algebra with separately continuous associative product, and  $\alpha: \mathbb{R}^n \times \mathcal{A} \rightarrow \mathcal{A}$  a smooth, polynomially bounded  $\mathbb{R}^n$ -action by automorphisms.*

- i.) If  $\mathcal{A}$  has an identity 1, this is also an identity for the deformed product (4.11).  
ii.) If  $\mathcal{A}$  is a  $*$ -algebra with continuous  $*$ -involution and  $\theta$  is skew-symmetric with respect to the inner product used in the oscillatory integrals defining the deformed product, then  $a \mapsto a^*$  is also a star involution for the deformed product, i.e.,

$$(a \times_{\theta} b)^* = b^* \times_{\theta} a^*, \quad a, b \in \mathcal{A}. \quad (4.15)$$

Proof. The first part is clear: since  $\alpha$  acts by automorphisms, we have  $\alpha_x(1) = 1$  for all  $x \in \mathbb{R}^n$ . Hence, by Lemma 4.8, i.),

$$a \times_{\theta} 1 = a1 = a, \quad 1 \times_{\theta} a = 1a = a$$

for any  $a \in \mathcal{A}$ . For the second part, we note that as the involution  $a \mapsto a^*$  is antilinear and continuous, we can use (3.20) and Lemma 3.7 to compute for  $a, b \in \mathcal{A}$

$$\begin{aligned} (a \times_{\theta} b)^* &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} (\alpha_{-\theta p} a \alpha_x b)^* \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \alpha_x b^* \alpha_{-\theta p} a^* \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dx e^{i\langle p, x \rangle} \alpha_{-\theta^T x} b^* \alpha_p a^* \\ &= b^* \times_{-\theta^T} a^*. \end{aligned}$$

In case  $\theta$  is skew-symmetric, i.e.,  $\theta^T = -\theta$ , (4.15) follows.  $\square$

Again, these statements are well-known in Rieffel's setting [23]. Analogous to the preceding proposition, there exist two closely related properties in the module deformation setting of Theorem 4.9: First, if a vector  $\psi \in \mathcal{E}$  is  $\beta$ -invariant, then we have  $a_{\theta}\psi = a\psi$ . This is again a straightforward consequence of Lemma 4.8, i.).

Second, in the case of a covariant Hilbert space representation of a  $*$ -algebra  $\mathcal{A}$ , we have a compatability between the  $*$ -operation and the deformation similar to Proposition 4.10 ii.). To describe this, consider a locally convex sequentially complete  $*$ -algebra  $\mathcal{A}$  with a smooth polynomially bounded  $\mathbb{R}^n$ -action  $\alpha^{\mathcal{A}}$  by  $*$ -automorphisms. Let furthermore  $\mathcal{H}$  be a Hilbert space carrying a strongly continuous unitary representation  $u$  of  $\mathbb{R}^n$ , and let  $\mathcal{E} \subset \mathcal{H}$  denote the subspace of smooth vectors for  $u$ . We consider a covariant representation of  $\mathcal{A}$ , i.e. a  $*$ -representation  $\pi$  of  $\mathcal{A}$  by (closable) operators defined on  $\mathcal{E}$  such that  $\pi(\alpha_x^{\mathcal{A}}(a))\psi = u(x)\pi(a)u(x)^{-1}\psi$  for all  $a \in \mathcal{A}$ ,  $x \in \mathbb{R}^n$ ,  $\psi \in \mathcal{E}$ .

Then we can apply our deformation formula to the module map  $\mu(a, \psi) := \pi(a)\psi$ . In case of a skew-symmetric deformation parameter  $\theta$ , the map  $\pi_{\theta}$  defined by the deformed module map,  $\pi_{\theta}(a)\psi := \mu_{\theta}(a, \psi)$ , then gives a  $*$ -representation of  $\mathcal{A}_{\theta}$  on  $\mathcal{E}$ , i.e.

$$\pi_{\theta}(a^*)\psi = \pi(a)^*\psi, \quad a \in \mathcal{A}, \psi \in \mathcal{E}. \quad (4.16)$$

In a  $C^*$ -framework with order 0 actions, this fact has been established in [8, Lemma 2.2]. Since the proof is essentially the same in the present situation, we refrain from giving the details here.

## 5 Examples

In this section we present a number of explicit examples of polynomially bounded  $\mathbb{R}^n$ -actions complying with the conditions in Definition 4.1. In particular, we show how target spaces with unbounded orders appear naturally when studying compactly supported  $\mathbb{R}^n$ -actions.

### 5.1 The canonical $\mathbb{R}^n$ -action on symbol spaces

The first example is the action studied in Section 2.3 on the symbol spaces.

**Proposition 5.1** *Let  $V$  be a sequentially complete locally convex space with a defining system of seminorms  $\mathcal{Q}$ , and let  $\mathbf{m}, \boldsymbol{\rho}$  be an order and a type for  $\mathcal{Q}$ , with  $\boldsymbol{\rho} \geq 0$ . Then  $(\alpha_x F)(y) := F(x + y)$  is a smooth polynomially bounded  $\mathbb{R}^n$ -action on  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  of order  $\hat{\mathbf{m}}(\|\cdot\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}) := |\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})| \mu|$  and type  $\hat{\boldsymbol{\rho}}(\|\cdot\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}) := 0$  where  $\mathbf{q} \in \mathcal{Q}$ ,  $\mu \in \mathbb{N}_0^n$ , as defined in Definition 4.1. More precisely, for any  $\mathbf{q} \in \mathcal{Q}$ ,  $\mu \in \mathbb{N}_0^n$ , there exists  $C_{\mathbf{q}, \mu} > 0$  such that*

$$\|\alpha(F)\|_{\|\cdot\|_{\mathbf{q}, \mu}^{\hat{\mathbf{m}}, 0}, \nu} \leq C_{\mathbf{q}, \mu} \|F\|_{\mathbf{q}, \mu + \nu}^{\mathbf{m}, \boldsymbol{\rho}} \quad (5.1)$$

for all  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ ,  $\nu \in \mathbb{N}_0^n$ .

Proof. It has been shown in Proposition 2.24 that  $\alpha$  is an  $\mathbb{R}^n$ -action on  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ , and in Proposition 2.27 that  $\mathbb{R}^n \ni x \mapsto \alpha_x(F) \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  is smooth for each  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  if  $\boldsymbol{\rho} \geq 0$ . To derive the statements about the polynomial bounds, let  $\mathbf{q} \in \mathcal{Q}$ ,  $\mu, \nu \in \mathbb{N}_0^n$ , and  $F \in S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  with  $\boldsymbol{\rho} \geq 0$ . The derivatives  $\partial_x^\nu \alpha_x(F) = \alpha_x(\partial_x^\nu F)$ , see (2.61), satisfy according to Lemma 2.22

$$\|\partial_x^\nu \alpha_x(F)\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} = \|\alpha_x(\partial_x^\nu F)\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} \leq c(x) \|\partial_x^\nu F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}$$

with a positive scalar symbol  $c \in S^{|\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})| \mu|, 1}(\mathbb{R}^n, \mathbb{R})$ . Furthermore, we have by application of Proposition 2.4 and Proposition 2.3, *iv.*)

$$\|\partial_x^\nu F\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}} = \|F\|_{\mathbf{q}, \mu + \nu}^{\mathbf{m} + \boldsymbol{\rho}|\nu|, \boldsymbol{\rho}} \leq \|F\|_{\mathbf{q}, \mu + \nu}^{\mathbf{m}, \boldsymbol{\rho}}$$

since  $\boldsymbol{\rho} \geq 0$ . With these two bounds, we arrive at

$$\|\alpha(F)\|_{\|\cdot\|_{\mathbf{q}, \mu}^{\hat{\mathbf{m}}, 0}, \nu} = \sup_{x \in \mathbb{R}^n} \frac{\|\partial_x^\nu \alpha_x(F)\|_{\mathbf{q}, \mu}^{\mathbf{m}, \boldsymbol{\rho}}}{(1 + \|x\|^2)^{\frac{1}{2} \hat{\mathbf{m}}}} \leq \sup_{x \in \mathbb{R}^n} \frac{c(x)}{(1 + \|x\|^2)^{\frac{1}{2} |\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})| \mu|}} \|F\|_{\mathbf{q}, \mu + \nu}^{\mathbf{m}, \boldsymbol{\rho}},$$

which establishes (5.1) with the constant  $C_{\mathbf{q}, \mu} := \|c\|_0^{|\mathbf{m}(\mathbf{q}) - \boldsymbol{\rho}(\mathbf{q})| \mu|} < \infty$ . Hence  $\alpha(F) \in S^{\hat{\mathbf{m}}, 0}(\mathbb{R}^n, S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V))$  and  $F \mapsto \alpha(F)$  is continuous, as required in Definition 4.1.  $\square$

By the same arguments, one checks that  $(\alpha_x(F))(y) = F(x + y)$  gives a smooth polynomially bounded  $\mathbb{R}^n$ -action on the vector valued Schwartz space  $\mathcal{S}(\mathbb{R}^n, V)$  (Definition 2.18), topologized by the seminorms  $q_{m, \mu}(\cdot) := \|\cdot\|_{\mathbf{q}, \mu}^{-m, 0}$ , with  $\mathbf{q} \in \mathcal{Q}$ ,  $m \in \mathbb{N}_0$ ,  $\mu \in \mathbb{N}_0^n$  (2.47). Here the order is  $\hat{\mathbf{m}}(q_{m, \mu}) = m$ , and again  $\hat{\boldsymbol{\rho}}(q_{m, \mu}) = 0$ .

**Remark 5.2** Remarkably, in both examples the orders  $\hat{\mathbf{m}}$  of the induced action is necessarily unbounded, even if we started with symbols of bounded order. Only in the particular case where  $\mathbf{m}(\mathbf{q}) = 0 = \boldsymbol{\rho}(\mathbf{q})$  we get again a bounded order  $\hat{\mathbf{m}} = 0$ . This was the particular case of an *isometric* action as discussed by Rieffel in [23].

If  $V = \mathcal{A}$  is an algebra with continuous product, we can use the action  $\alpha$  to deform the pointwise product in  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, \mathcal{A})$  as in (4.11), with some deformation parameter  $\theta \in \mathbb{R}^{n \times n}$ . As the evaluation maps  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, \mathcal{A}) \ni F \mapsto F(x) \in \mathcal{A}$  are continuous, we have the explicit formula

$$(F \times_{\theta}^{\alpha} G)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp dy e^{i\langle p, y \rangle} F(y + \theta p) G(y + x) \quad (5.2)$$

as a  $\mathcal{A}$ -valued oscillatory integral.

In addition to  $\alpha$ , we have on the Schwartz space also smooth polynomially bounded  $\mathbb{R}^n$ -actions of the form

$$(\beta_x F)(y) := e^{i\langle x, y \rangle} F(y), \quad (5.3)$$

where  $(\cdot, \cdot)$  denotes a bilinear form on  $\mathbb{R}^n$ . Considered on a symbol space  $S^{\mathbf{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$  of fixed order, these actions are not smooth, but on  $\mathcal{S}(\mathbb{R}^n, V)$ , they comply with Definition 4.1, with order  $\hat{\mathbf{m}}(\mathbf{q}_{m, \mu}) = |\mu|$  and type  $\hat{\boldsymbol{\rho}}(\mathbf{q}_{m, \mu}) = 0$ . Taking  $V = \mathcal{A}$  to be an algebra, the deformation of the pointwise product in  $\mathcal{S}(\mathbb{R}^n, \mathcal{A})$  with the action (5.3) is however almost trivial; one has  $(F \times_{\theta}^{\beta} G)(x) = e^{i\langle \theta A x, x \rangle} F(x) G(x)$  with a matrix  $A$  depending on the choice of inner product on  $\mathbb{R}^n$ .

We now explain how some deformations of algebras of scalar-valued functions discussed in the literature fit into our framework. The first and best-known example is clearly the scalar Schwartz space  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$  with pointwise product. Here the deformed product (5.2) even exists pointwise as a Riemann integral because of the decay of the integrand. It is usually referred to as Moyal product or twisted product, see e.g. [15].

Another version of this is to consider  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$  as an algebra with convolution  $(f * g)(x) = \int_{\mathbb{R}^n} dy f(y) g(x - y)$  as product, and the multiplicative action (5.3). Taking all inner products of  $\mathbb{R}^n$  as the usual Euclidean inner product, and  $\theta$  to be antisymmetric, we find

$$(f *_{\theta}^{\beta} g)(x) = \int_{\mathbb{R}^n} dy e^{i\langle x, \theta y \rangle} f(y) g(x - y). \quad (5.4)$$

This deformed product is usually referred to as a *twisted convolution* according to [15]. Since the Fourier transform  $\mathcal{F}: \mathcal{S} \rightarrow \mathcal{S}$  intertwines the pointwise product and convolution as well as the actions  $\alpha$  and  $\beta$ , the twisted convolution product is equivalent to the product  $\times_{\theta}^{\alpha}$ .

The deformed products  $\times_{\theta}^{\alpha}$  and  $*_{\theta}^{\beta}$  can be extended from  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$  to spaces of distributions [12, 15]. In particular, in [12] it is explained how  $\times_{\theta}^{\alpha}$  can be defined on the distribution space  $\mathcal{O}'_M(\mathbb{R}^n)$ , the dual of the space  $\mathcal{O}_M(\mathbb{R}^n)$  of tempered smooth functions. Recall that  $\mathcal{O}_M(\mathbb{R}^n)$  is defined as the set of all smooth  $f: \mathbb{R}^n \rightarrow \mathbb{C}$  such that for each multiindex  $\mu$ , there exists some  $k \in \mathbb{Z}$  such that  $x \mapsto (1 + \|x\|^2)^k |(\partial_x^{\mu} f)(x)|$  is bounded. In our notation, that is  $\mathcal{O}_M = \cup_{m, \rho} S^{m, \rho}(\mathbb{R}^n, \mathbb{C})$ , where the union runs over all orders

and types  $m, \rho \in \mathbb{R}$ . Similarly, the classical function space  $\mathcal{O}_C(\mathbb{R}^n)$  is in our notation  $\mathcal{O}_C(\mathbb{R}^n) = \cup_m S^{m,0}(\mathbb{R}^n, \mathbb{C}) = S^{\infty,0}(\mathbb{R}^n, \mathbb{C})$ . Clearly  $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{O}_C(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$  and  $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{O}'_M(\mathbb{R}^n) \subset \mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ , and the Fourier transform  $\mathcal{F}$  on  $\mathcal{S}'(\mathbb{R}^n)$  restricts to isomorphisms  $\mathcal{O}_C(\mathbb{R}^n) \rightarrow \mathcal{O}'_M(\mathbb{R}^n)$  and  $\mathcal{O}_M(\mathbb{R}^n) \rightarrow \mathcal{O}'_C(\mathbb{R}^n)$ . Since  $\mathcal{O}_C(\mathbb{R}^n)$  contains only symbols of type  $\rho = 0$ , we can form the deformed products  $f \times_\theta^\alpha g$  (5.2) for  $f, g \in \mathcal{O}_C(\mathbb{R}^n)$ . (For  $f, g \in \mathcal{O}_M(\mathbb{R}^n)$ , this is not possible because we need restrictions on the type for  $\alpha$  to be smooth and the oscillatory integrals to exist.) Making use of the Fourier transform  $\mathcal{F}: \mathcal{O}_C(\mathbb{R}^n) \rightarrow \mathcal{O}'_M(\mathbb{R}^n)$ , this also gives us a product on  $\mathcal{O}'_M(\mathbb{R}^n)$ ,

$$T \times S := \mathcal{F}(\mathcal{F}^{-1}T \times_\theta^\alpha \mathcal{F}^{-1}S). \quad (5.5)$$

As  $\mathcal{F}$  intertwines the actions  $\alpha$  and  $\beta$ , it is easy to see that (5.5) coincides with the “other twisted convolution” constructed in [12].

## 5.2 Compactly supported $\mathbb{R}^n$ -actions

In this subsection we construct and study a different class of smooth polynomially bounded  $\mathbb{R}^n$ -actions on function spaces. The actions we are interested in here are given by pullbacks of  $\mathbb{R}^n$ -actions  $\tau$  on  $\mathbb{R}^n$  which act non-trivially only in a compact set  $K$ , i.e. satisfy  $\tau_x(y) = y$  for all  $y \notin K$ ,  $x \in \mathbb{R}^n$ . We want to construct  $\tau$  in such a way that  $\alpha_x^K(f) := f \circ \tau_x$  is smooth and polynomially bounded in the sense of Definition 4.1, say on  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ . For simplicity, we restrict to scalar-valued functions here. It is clear that we cannot hope for an isometric action as required by Rieffel’s original construction as soon as we leave the  $C^0$ -framework: controlling also derivatives as needed in the  $C^\infty$ -topology will necessarily lead to a non-isometric action.

To check what kind of condition on  $\tau$  is necessary for this, consider a function  $f_j$  which coincides with a coordinate  $x \mapsto x_j$ ,  $j = 1, \dots, n$ , on  $K$ . If  $\alpha^K(f_j) \in S^{\hat{m},0}(\mathbb{R}^n, \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}))$  for some appropriate  $\hat{m}$ , then the supremum

$$\|\alpha^K(f_j)\|_{p_{K,l},\mu}^{\hat{m}(p_{K,l}),0} = \sup_{x \in \mathbb{R}^n} \frac{p_{K,l}(\partial_x^\mu \alpha_x^K f_j)}{(1 + \|x\|^2)^{\frac{1}{2}\hat{m}(p_{K,l})}} = \sup_{\substack{x \in \mathbb{R}^n \\ y \in K, |\nu| \leq l}} \frac{|\partial_x^\mu \partial_y^\nu \tau_x(y)_j|}{(1 + \|x\|^2)^{\frac{1}{2}\hat{m}(p_{K,l})}} \quad (5.6)$$

must be finite. Hence we need bounds of the form  $|\partial_y^\nu \partial_x^\mu \tau_x(y)_j| \leq c_{\mu l}(1 + \|x\|^2)^{\frac{1}{2}b_l}$  for all  $\nu \in \mathbb{N}_0^n$  with  $|\nu| \leq l$ . Taking into account that  $\tau$  satisfies  $\tau_x(K) = K$  for all  $x \in \mathbb{R}^n$  by its support property, that  $K$  is compact, and that  $\tau$  is an  $\mathbb{R}^n$ -action, it follows that we can choose  $b_0 = 0$ . These observations motivate the following definition.

**Definition 5.3** *Let  $K \subset \mathbb{R}^n$  be compact, and  $\mathbf{b} := \{b_l\}_{l \in \mathbb{N}_0} \subset \mathbb{R}_+$  a sequence starting with  $b_0 = 0$ . A smooth  $\mathbb{R}^n$ -action with support in  $K$  and order  $\mathbf{b}$  is a smooth function  $\tau: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that*

- i.)  $\tau_x(\tau_{x'}(y)) = \tau_{x+x'}(y)$  for all  $x, x', y \in \mathbb{R}^n$ .
- ii.)  $\tau_x(y) = y$  for all  $x \in \mathbb{R}^n$  and all  $y \in \mathbb{R}^n \setminus K$ .
- iii.) For each  $\mu \in \mathbb{N}_0^n$ ,  $l \in \mathbb{N}_0$ , there exists a constant  $c_{l\mu} > 0$  such that

$$\sup_{\substack{y \in K, |\nu| \leq l \\ j \in \{1, \dots, n\}}} |\partial_y^\nu \partial_x^\mu \tau_x(y)_j| \leq c_{l\mu}(1 + \|x\|^2)^{\frac{1}{2}b_l} \quad (5.7)$$

holds for all  $x \in \mathbb{R}^n$ .

We will later construct explicit examples of actions satisfying these assumptions. Postponing this construction for a moment, we first show that such  $\tau$  do indeed define smooth polynomially bounded  $\mathbb{R}^n$ -actions by pullback. To begin with, we note the following elementary lemma.

**Lemma 5.4** *Let  $\tau$  be a smooth  $\mathbb{R}^n$ -action with support in a compact set  $K \subset \mathbb{R}^n$ , and order  $\mathbf{b}$ . Then for each  $\mu \in \mathbb{N}_0^n$ ,  $l \in \mathbb{N}_0$ , there exists a constant  $C_{l\mu} > 0$  such that*

$$\sup_{y \in K, |\nu| \leq l} |\partial_y^\nu \partial_x^\mu f(\tau_x(y))| \leq C_{l\mu} (1 + \|x\|^2)^{\frac{1}{2}(b_1 + \dots + b_l)} \cdot p_{K, l+|\mu|}(f) \quad (5.8)$$

for all  $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ ,  $x \in \mathbb{R}^n$ . Here  $p_{K, l+|\mu|}$  denotes the usual  $\mathcal{C}^\infty$ -seminorms (2.7) with  $q = |\cdot|$ .

Proof. We first consider the case  $l = 0$  without derivatives with respect to  $y$ . By the chain rule, we have

$$\partial_x^\mu f(\tau_x(y)) = \sum_{\lambda \leq \mu} (\partial^\lambda f)(\tau_x(y)) \cdot g_\lambda(x, y),$$

where the  $g_\lambda(x, y)$  are polynomials in partial derivatives of the  $\tau_x(y)_j$  with respect to the components of  $x$ . According to (5.7) with  $l = 0$  (and  $b_0 = 0$ ), these functions are uniformly bounded in  $x \in \mathbb{R}^n$  and  $y \in K$ . Furthermore, we have  $\tau_x(y) \in K$  for all  $x \in \mathbb{R}^n$ , since  $y \in K$ . Hence (5.8) follows by straightforward estimate. We now proceed by induction and assume that (5.8) holds for some  $l \in \mathbb{N}_0^n$ , and all  $\mu \in \mathbb{N}_0^n$ ,  $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ . Then,  $j \in \{1, \dots, n\}$ ,  $|\nu| \leq l$ ,

$$\begin{aligned} \partial_y^{\nu+e_j} \partial_x^\mu f(\tau_x(y)) &= \partial_y^\nu \partial_x^\mu \sum_{j'=1}^n \partial_y^{e_j} \tau_x(y)_{j'} \cdot (\partial^{e_{j'}} f)(\tau_x(y)) \\ &= \sum_{j'=1}^n \sum_{\substack{\nu' \leq \nu \\ \mu' \leq \mu}} \binom{\nu}{\nu'} \binom{\mu}{\mu'} \left( \partial_y^{\nu-\nu'+e_j} \partial_x^{\mu-\mu'} \tau_x(y)_{j'} \right) \left( \partial_y^{\nu'} \partial_x^{\mu'} (\partial^{e_{j'}} f)(\tau_x(y)) \right). \end{aligned}$$

In this sum, the derivatives of  $\tau_x(y)_{j'}$  can be estimated directly with (5.7), taking into account  $|\nu - \nu' + e_j| \leq l + 1$ . For the derivatives of  $f$ , we can use (5.8) by our induction hypothesis, since  $|\nu'| \leq |\nu| \leq l$ . This yields constants  $C_{j'\nu'\mu'} > 0$  such that

$$\begin{aligned} |\partial_y^{\nu+e_j} \partial_x^\mu f(\tau_x(y))| &\leq \sum_{j', \nu', \mu'} C_{j'\nu'\mu'} (1 + \|x\|^2)^{\frac{1}{2}b_{l+1}} (1 + \|x\|^2)^{\frac{1}{2}(b_1 + \dots + b_l)} p_{K, |\mu'|+|\nu'|}(\partial^{e_{j'}} f) \\ &\leq C'_{j\mu\nu} (1 + \|x\|^2)^{\frac{1}{2}(b_1 + \dots + b_l + b_{l+1})} \cdot p_{K, l+1+|\mu|}(f). \end{aligned}$$

Since  $j$  was arbitrary, (5.8) follows by induction in  $l$ . □

**Proposition 5.5** *Let  $\tau$  be a smooth  $\mathbb{R}^n$ -action on  $\mathbb{R}^n$ , with order  $\mathbf{b}$  and support in some compact set  $K \subset \mathbb{R}^n$ . Then its pullback  $(\alpha_x^K f)(y) := f(\tau_x(y))$  is a smooth polynomially bounded  $\mathbb{R}^n$ -action on  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ , on each symbol space  $S^{m,\rho}(\mathbb{R}^n, \mathbb{C})$ ,  $m, \rho \in \mathbb{R}$ , and on the Schwartz space  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ .*

Proof. Let us first consider  $\alpha^K$  acting on  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ . It is clear that it is an  $\mathbb{R}^n$ -action on this space because  $\tau$  is an action and smooth. To estimate its seminorms, let  $J \subset \mathbb{R}^n$  be compact,  $\mu, \nu \in \mathbb{N}_0^n$ , and  $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ . Taking into account that  $\tau$  acts trivially outside  $K$ , and using the bounds of Lemma 5.4, we find

$$\begin{aligned} & \sup_{\substack{x \in \mathbb{R}^n \\ y \in J, |\nu| \leq l}} \frac{|\partial_y^\nu \partial_x^\mu f(\tau_x(y))|}{(1 + \|x\|^2)^{\frac{1}{2}(b_1 + \dots + b_l)}} \\ & \leq \sup_{\substack{x \in \mathbb{R}^n \\ y \in J \setminus K, |\nu| \leq l}} \frac{|\partial_y^\nu \partial_x^\mu f(y)|}{(1 + \|x\|^2)^{\frac{1}{2}(b_1 + \dots + b_l)}} + \sup_{\substack{x \in \mathbb{R}^n \\ y \in J \cap K, |\nu| \leq l}} \frac{|\partial_y^\nu \partial_x^\mu f(\tau_x(y))|}{(1 + \|x\|^2)^{\frac{1}{2}(b_1 + \dots + b_l)}} \\ & \leq \delta_{\mu,0} \sup_{\substack{x \in \mathbb{R}^n \\ y \notin J \setminus K, |\nu| \leq l}} \frac{|\partial_y^\nu f(y)|}{(1 + \|x\|^2)^{\frac{1}{2}(b_1 + \dots + b_l)}} + C_{l\mu} p_{K,l+|\mu|}(f) \\ & = \delta_{\mu,0} p_{J \setminus K,l}(f) + C_{l\mu} p_{K,l+|\mu|}(f), \end{aligned}$$

where the last step relies on  $b_l \geq 0$ . This estimate shows in particular that for any  $x \in \mathbb{R}^n$ , the map  $\alpha_x^K: \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$  is continuous, a fact which is known to be true for *any* diffeomorphism. Moreover, once we have checked that  $x \mapsto \alpha_x^K f$  is smooth in the topology of  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ , the estimate also shows that  $\alpha^K(f)$  is a symbol in  $S^{\hat{\mathbf{m}},0}(\mathbb{R}^n, \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}))$ , of order  $\hat{\mathbf{m}}(p_{J,l}) := b_1 + \dots + b_l$ , and that  $f \mapsto \alpha^K(f)$  is continuous. So in order to verify all conditions of Definition 4.1, it only remains to establish the smoothness of  $\alpha^K$ : but this is true for arbitrary smooth Lie group actions on smooth manifolds. We now consider  $\alpha^K$  on the symbol and Schwartz subspaces of  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ . Since  $\tau$  acts non-trivially only in a compact set, it is clear that these subspaces are invariant under  $\alpha^K$ , and  $\alpha^K$  restricts to  $\mathbb{R}^n$ -actions on all these spaces. Concerning smoothness, note that the functions  $\alpha_x^K(f) - f$  and  $\varepsilon^{-1}(\alpha_{\varepsilon e_j}^K(f) - f) - \partial_t \alpha_{te_j}^K(f)|_{t=0}$  have compact support in  $K$  for all  $f \in S^{m,\rho}$ ,  $\varepsilon > 0$ ,  $j \in \{1, \dots, n\}$ ,  $x \in \mathbb{R}^n$ . So their symbol seminorms  $\|\cdot\|_\nu^{m,\rho}$  can be estimated against some  $p_{K,l}(\cdot)$ . But the latter seminorms converge to zero for  $x \rightarrow 0$  respectively  $\varepsilon \rightarrow 0$ , by the preceding results about  $\alpha^K$  on  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ . Thus we conclude that  $\alpha^K$  is also smooth on the symbol spaces  $S^{m,\rho}(\mathbb{R}^n, \mathbb{C})$ ,  $m, \rho \in \mathbb{R}$ , and the Schwartz space  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ . The symbol property of  $\alpha^K(f)$ , and the continuity of  $f \mapsto \alpha^K(f)$  for these spaces can now be estimated as before, by splitting  $f = f_0 + f_1 \in S^{m,\rho}(\mathbb{R}^n, \mathbb{C})$  into a compactly supported symbol  $f_0$ , and a symbol  $f_1$  with support disjoint from  $K$  which is fixed by  $\alpha^K$ .  $\square$

We now turn to the construction of examples of smooth compactly supported  $\mathbb{R}^n$ -actions, and consider the one-dimensional case  $n = 1$  first. As a starting point, we use the same idea as in [19] and take a diffeomorphism  $\gamma: (-1, 1) \rightarrow \mathbb{R}$  to define

$$\tau_x(y) := \begin{cases} \gamma^{-1}(\gamma(y) + x) & ; \quad |y| < 1 \\ y & ; \quad |y| \geq 1 \end{cases}, \quad x \in \mathbb{R}. \quad (5.9)$$



It is clear that  $\tau$  is an  $\mathbb{R}$ -action, *i.e.*,  $\tau_x(\tau_{x'}(y)) = \tau_{x+x'}(y)$  for all  $x, x', y \in \mathbb{R}$ , and  $\tau$  acts non-trivially only inside the interval  $K := [-1, 1]$ . But we have to choose  $\gamma$  in such a way that also the smoothness and boundedness assumptions of Definition 5.3 are satisfied. For the discussion of these two properties, it is instructive to view  $\tau$  as the flow of an autonomous ordinary differential equation  $d\phi/dx = L(\phi(x))$  with initial condition  $\phi(0) = y$ . Differentiation of  $\phi(x) = \tau_x(y)$  (5.9) with respect to  $x$  then shows that

$$L(x) = \begin{cases} \frac{1}{\gamma'(x)} & ; \quad |x| < 1 \\ 0 & ; \quad |x| \geq 1 \end{cases}. \quad (5.10)$$

It is a well-known fact that the solutions  $x \mapsto \phi(x) = \tau_x(y)$  will depend smoothly on  $x$  and the initial condition  $y$  if  $L$  is smooth. Thus smoothness of  $\tau$  is guaranteed if  $\gamma'(x)$  diverges fast enough as  $x \rightarrow \pm 1$ , such that (5.10) is smooth. On the other hand, the bounds on  $\partial_x^k \partial_y^l \tau_x(y)$  that can be obtained by exploiting that  $\tau$  is the flow of a differential equation with compactly supported  $L$  are only of *exponential* type. Therefore, we show in the following lemma that by a careful adjustment of the diffeomorphism  $\gamma$ , one can achieve *polynomial* bounds on  $\partial_x^k \partial_y^l \tau_x(y)$ .

**Lemma 5.6** *There exist smooth  $\mathbb{R}$ -actions on  $\mathbb{R}$  with support in  $[-1, 1]$  and order  $b_l = 2l + 1$ , which act transitively on  $(-1, 1)$ .*

Proof. The action will be constructed in the form (5.9) with appropriately chosen  $\gamma$ . It is already clear from (5.9) that  $\tau$  is an action with support in  $[-1, 1]$ , acting transitively on  $(-1, 1)$ . To verify the crucial bounds (5.7),

$$\sup_{|y| \leq 1} |\partial_x^k \partial_y^l \tau_x(y)| \leq c_{lk} (1 + x^2)^{\frac{1}{2}(2l+1)}, \quad x \in \mathbb{R}, \quad (5.11)$$

we first derive a formula for the derivatives of  $\tau_x(y) = \gamma^{-1}(\gamma(y) + x)$ ,  $|y| < 1$ , for generic diffeomorphisms  $\gamma$ . This is done with the help of two differentiation identities, the first of which states that multiple derivatives of  $\gamma^{-1}$  have the form

$$\partial_y^l \gamma^{-1}(y) = \sum_n c_n \frac{\gamma'(\gamma^{-1}(y))^{n_1} \cdots \gamma^{(l)}(\gamma^{-1}(y))^{n_l}}{\gamma'(\gamma^{-1}(y))^{2l-1}}, \quad (5.12)$$

where the sum runs over finitely many terms with numerical coefficients  $c_n$ . In the above formula, the powers  $n_j$  satisfy  $n_1 + \cdots + n_l = l - 1$  in each term, a fact that can easily be proven by induction in  $l$ .

The second identity is an iterated chain rule for smooth functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\partial_y^l (f(g(y))) = \sum_{r=1}^l c'_r g'(y)^{s_1} \cdots g^{(l)}(y)^{s_l} \cdot f^{(r)}(g(y)), \quad (5.13)$$

where the  $c'_r$  are numerical coefficients and the powers  $s_j$  satisfy  $s_1 + \cdots + s_l = r$  in each term. Again, the proof by induction is straightforward.



Application of these two differentiation rules to  $\tau_x(y) = \gamma^{-1}(\gamma(y) + x)$ ,  $|y| < 1$ , yields

$$\begin{aligned}
\partial_x^k \partial_y^l \tau_x(y) &= \partial_x^k \sum_{r=1}^l c'_r \gamma'(y)^{s_1} \dots \gamma^{(l)}(y)^{s_l} \cdot (\gamma^{-1})^{(r)}(\gamma(y) + x) \\
&= \sum_{r=1}^l c'_r \gamma'(y)^{s_1} \dots \gamma^{(l)}(y)^{s_l} \cdot (\gamma^{-1})^{(r+k)}(\gamma(y) + x) \\
&= \sum_{r=1}^l c'_r \sum_n c_n \frac{\gamma'(y)^{s_1} \dots \gamma^{(l)}(y)^{s_l} \cdot \gamma'(\tau_x(y))^{n_1} \dots \gamma^{(r+k)}(\tau_x(y))^{n_{r+k}}}{\gamma'(\tau_x(y))^{2(r+k)-1}}. \quad (5.14)
\end{aligned}$$

To obtain useful bounds on this expression, we have to estimate the higher derivatives  $|\gamma^{(m)}(y)|$  for  $m = 1, \dots, l$ , and  $|\gamma^{(j)}(\tau_x(y))|$  for  $j = 1, \dots, r + k$ , in terms of the first derivatives  $|\gamma'(\tau_x(y))|$ . Simple estimates of this form do apparently not exist for generic  $\gamma$ . We therefore choose  $\gamma$  of a special form, which will allow for convenient computations.

So let  $\gamma$  be antisymmetric, *i.e.*,  $\gamma(-y) = -\gamma(y)$ , and choose it to be equal to  $e(y) := \exp \frac{1}{1-y}$  for  $y \geq \frac{1}{2}$ . Note that this choice already implies that  $\tau: \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth, since all derivatives of  $L(y) := 1/\gamma'(y) = (1-y)^2 e^{-1/(1-y)}$  converge to zero for  $y \rightarrow 1$ . Moreover, a short calculation shows that the tangent of  $\gamma$  in  $y = \frac{1}{2}$  has its zero at  $y = \frac{1}{4}$  and consequently, we can choose  $\gamma$  on  $[-\frac{1}{2}, \frac{1}{2}]$  in such a way that  $\gamma'(y)$  increases monotonically as  $|y|$  increases. In particular, we then have the lower bound  $\gamma'(y) \geq \gamma'(0) > 0$ ,  $|y| < 1$ .

We now turn to estimating the derivatives  $|\gamma^{(j)}(y)|$  for a diffeomorphism  $\gamma$  with the specified properties. Since  $\gamma'$  is bounded from below and  $\gamma^{(j)}$  is continuous, there exist constants  $C_j < \infty$  such that  $|\gamma^{(j)}(y)/\gamma'(y)| \leq C_j$  for all  $y \in [-\frac{1}{2}, \frac{1}{2}]$ . For  $y > \frac{1}{2}$ , we can use the explicit form  $\gamma(y) = \exp \frac{1}{1-y}$ , which implies that  $\gamma^{(j)}(y)$  is the product of  $\gamma(y)$  and a polynomial of order  $2j$  in  $\frac{1}{1-y}$ . Hence  $\gamma^{(j)}(y)/\gamma'(y)$  coincides for  $y > \frac{1}{2}$  with a rational function of  $y$  which diverges polynomially as  $y \rightarrow 1$ . Because of the symmetry properties of  $\gamma'$  and  $\gamma^{(j)}$ , the same is true for the region  $y < -\frac{1}{2}$  and the limit  $y \rightarrow -1$ . But as  $\gamma'(y)$  has no zeros and diverges exponentially for  $|y| \rightarrow \pm 1$ , we find for any  $\varepsilon > 0$  a constant  $C_{j,\varepsilon}$  such that

$$\frac{|\gamma^{(j)}(y)|}{\gamma'(y)} \leq C_{j,\varepsilon} \gamma'(y)^\varepsilon, \quad y \in (-1, 1). \quad (5.15)$$

Since  $\tau_x$  leaves the interval  $(-1, 1)$  invariant for any  $x \in \mathbb{R}$ , we also have

$$|\gamma^{(j)}(\tau_x(y))| \leq C_{j,\varepsilon} \gamma'(\tau_x(y))^{1+\varepsilon}, \quad x \in \mathbb{R}, y \in (-1, 1). \quad (5.16)$$

Next we derive a bound on the ratios  $\frac{\gamma'(y)}{\gamma'(\tau_x(y))}$ . For this it is sufficient to consider  $y \in [0, 1)$  because of the symmetry of  $\gamma'$ , and for fixed  $x \in \mathbb{R}$ , we split this interval at

$$\xi(x) := \tau_{2|x|}(\frac{1}{2}) = e^{-1}(e(\frac{1}{2}) + 2|x|) \geq \frac{1}{2}, \quad (5.17)$$

and estimate in the two regions  $y \in [0, \xi(x)]$  and  $y \in (\xi(x), 1)$  separately. Note that  $e^{-1}(y) = 1 - 1/\log y$  and  $e'(y) = (1-y)^{-2}e(y)$ .

In the inner region  $[0, \xi(x))$ , the monotonicity of  $\gamma'$  and the explicit form of our diffeomorphism around  $\xi(x) \geq \frac{1}{2}$  yield

$$\begin{aligned} \frac{\gamma'(y)}{\gamma'(\tau_x(y))} &\leq \frac{\gamma'(\xi(x))}{\gamma'(0)} = \frac{e(\xi(x))}{\gamma'(0)(1-\xi(x))^2}, & |y| \leq \xi(x), \\ &= \gamma'(0)^{-1} (e(\frac{1}{2}) + 2|x|) \cdot \log(e(\frac{1}{2}) + 2|x|) \\ &\leq c(1+x^2)^\delta, \end{aligned} \quad (5.18)$$

where the power  $\delta > \frac{1}{2}$  can be chosen arbitrarily close to  $\frac{1}{2}$  (it will be fixed at the end of the proof), and the numerical constant  $c$  depends on  $\delta$ .

For the estimate in the outer region  $(\xi(x), 1)$ , we use the inequalities

$$e(y) + x \geq e(\xi(x)) - |x| = e(\frac{1}{2}) + |x| \geq e(\frac{1}{2}), \quad y \in [\xi(x), 1), \quad (5.19)$$

and  $e(y) \geq e(\xi(x)) > 2|x|$ , implying  $e(y) - |x| \geq \frac{1}{2}e(y) > 1$  for  $y > \xi(x)$ . It follows from (5.19) that  $\tau_x(y) = e^{-1}(e(y) + x)$  in this region. The explicit form of  $\gamma$  and these inequalities lead to a uniform bound on  $\frac{\gamma'(y)}{\gamma'(\tau_x(y))}$ ,

$$\begin{aligned} \frac{\gamma'(y)}{\gamma'(\tau_x(y))} &= \frac{1}{(1-y)^2 \log(e(y) + x)^2} \frac{e(y)}{e(y) + x}, & y \in (\xi(x), 1), \\ &\leq \frac{1}{(1-y)^2 \log(e(y) - |x|)^2} \frac{e(y)}{e(y) - |x|}. \\ &\leq \frac{2}{(1-y)^2 (\frac{1}{1-y} - \log 2)^2} \leq \frac{2}{(1 - \frac{1}{2} \log 2)^2}. \end{aligned} \quad (5.20)$$

After a possible readjustment of the constant  $c$  in (5.18) we therefore obtain

$$\gamma'(y) \leq c(1+x^2)^\delta \gamma'(\tau_x(y)), \quad x \in \mathbb{R}, y \in (-1, 1), \quad (5.21)$$

where  $\delta > \frac{1}{2}$  can still be chosen. Combining this bound with (5.15), we also have

$$|\gamma^{(m)}(y)| = \frac{|\gamma^{(m)}(y)|}{\gamma'(y)} \cdot \gamma'(y) \leq C'_{m,\varepsilon} (1+x^2)^{\delta(1+\varepsilon)} \cdot \gamma'(\tau_x(y))^{1+\varepsilon}. \quad (5.22)$$

We can now apply (5.16) and (5.22) to estimate (5.14). Taking into account that in each term in that sum, we have  $s_1 + \dots + s_l = r$  and  $n_1 + \dots + n_{r+k} = r+k-1$ , we get after collecting all factors

$$|\partial_x^k \partial_y^l \tau_x(y)| \leq \sum_n \sum_{r=1}^l C(1+x^2)^{\delta(1+\varepsilon)r} \gamma'(\tau_x(y))^{\varepsilon(2r+k-1)-k}, \quad (5.23)$$

where  $C$  represents all the numerical constants appearing in the various bounds, depending on  $l, k, m, n$  and some arbitrary  $\varepsilon > 0$ ,  $\delta > \frac{1}{2}$ . For  $k \geq 1$ , the exponent of  $\gamma'(\tau_x(y))$  is negative for sufficiently small  $\varepsilon$ , and then

$$|\partial_x^k \partial_y^l \tau_x(y)| \leq \sum_n \sum_{r=1}^l C(1+x^2)^{\delta(1+\varepsilon)r} \gamma'(0)^{\varepsilon(2r+k-1)-k} \leq C'(1+x^2)^{\delta(1+\varepsilon)l}, \quad |y| < 1.$$

For the case  $k = 0$ , note that since  $\tau_0(y) = y$  and  $l \geq 1$ , we have  $\partial_y^l \tau_0(y) = 0$  and can estimate via

$$|\partial_y^l \tau_x(y)| = \left| \int_0^x dx' \partial_{x'} \partial_y^l \tau_{x'}(y) \right| \leq |x| \cdot C'(1+x^2)^{\delta(1+\varepsilon)l} \leq C''(1+x^2)^{\delta(1+\varepsilon)l+\frac{1}{2}}. \quad (5.24)$$

Choosing  $\delta = \frac{2}{3}$  and  $\varepsilon = \frac{1}{2}$  now gives the claimed bounds (5.11).  $\square$

Next we show how the above constructed  $\mathbb{R}$ -action on  $\mathbb{R}$  can be promoted to suitable  $\mathbb{R}^n$ -action on  $\mathbb{R}^n$  based on the ideas from [24, Ex. 4.5].

**Lemma 5.7** *Let  $\tau^1$  be a smooth polynomially bounded  $\mathbb{R}$ -action on  $\mathbb{R}$  with support in  $[-1, 1]$ , as constructed in Lemma 5.6. Furthermore, let  $\varepsilon > 0$  and  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function which is equal to 1 on  $[-1, 1]$ , and has support in  $[-1 - \varepsilon, 1 + \varepsilon]$ . Then*

$$\tau_x^n(y) := (\tau_{x_1 \cdot \chi(y_1) \cdots \chi(y_n)}^1(y_1), \dots, \tau_{x_n \cdot \chi(y_1) \cdots \chi(y_n)}^1(y_n)) \quad (5.25)$$

*is a smooth polynomially bounded  $\mathbb{R}^n$ -action on  $\mathbb{R}^n$ , with support in  $[-1 - \varepsilon, 1 + \varepsilon]^{\times n}$ .*

Proof. Let  $I := [-1, 1]$  and  $I_\varepsilon := [-1 - \varepsilon, 1 + \varepsilon]$ . If  $y \notin I_\varepsilon^{\times n}$ , there exists  $j \in \{1, \dots, n\}$  such that  $y_j \notin I_\varepsilon$ , and hence  $\chi(y_1) \cdots \chi(y_n) = 0$ . Thus  $\tau_x^n(y) = (\tau_0^1(y_1), \dots, \tau_0^1(y_n)) = y$  in this case, which shows that  $\tau^n$  has support in  $B$ . As a composition of smooth functions  $\tau^n$  is smooth. To show that it is an action, let  $x, x', y \in \mathbb{R}^n$ ,  $j \in \{1, \dots, n\}$ , and compute

$$\begin{aligned} \tau_x^n(\tau_{x'}^n(y))_j &= \tau_{x_j \cdot \chi(\tau_{x'}^n(y)_1) \cdots \chi(\tau_{x'}^n(y)_n)}^1(\tau_{x'}^n(y)_j) \\ &= \tau_{x_j \cdot \chi(\tau_{x'}^1(y_1) \cdots \chi(y_n))}^1(\tau_{x'_j \cdot \chi(y_1) \cdots \chi(y_n)}^1(y_j)) \\ &= \tau_{x_j \cdot \chi(\tau_{x'_1 \chi(y_1) \cdots \chi(y_n)}^1(y_1) \cdots \chi(\tau_{x'_n \chi(y_1) \cdots \chi(y_n)}^1(y_n)) + x'_j \cdot \chi(y_1) \cdots \chi(y_n))}^1(y_j). \end{aligned}$$

This coincides with  $\tau_{x+x'}^n(y)_j = \tau_{(x_j+x'_j)\chi(y_1) \cdots \chi(y_n)}^1(y_j)$  if

$$\chi(y_1) \cdots \chi(y_n) = \chi(\tau_{x'_1 \chi(y_1) \cdots \chi(y_n)}^1(y_1)) \cdots \chi(\tau_{x'_n \chi(y_1) \cdots \chi(y_n)}^1(y_n)). \quad (5.26)$$

Assume some component  $y_k$  does not lie in  $I$ . Then  $\tau_{x'_k \chi(y_1) \cdots \chi(y_n)}^1(y_k) = y_k$  by the support properties of  $\tau^1$ . If, on the other hand,  $y_k \in I$ , then  $\tau_{x'_k \chi(y_1) \cdots \chi(y_n)}^1(y_k) \in I$  as well, and since  $\chi = 1$  on  $I$ , we find also in this case  $\chi(y_k) = \chi(\tau_{x'_k \chi(y_1) \cdots \chi(y_n)}^1(y_k))$ . Hence (5.26) holds for all  $x, y, y'$ , and it follows that  $\tau^n$  is an  $\mathbb{R}^n$ -action.

It remains to verify the bounds (5.7) on the derivatives of  $\tau^n$ , i.e. we have to estimate  $\partial_x^\mu \partial_y^\nu \tau_{x_j \chi(y_1) \cdots \chi(y_n)}^1(y_j)$ . In comparison to  $\partial_x^\mu \partial_y^\nu \tau_{x_j}^1(y_j)$ , the  $y$ -derivatives produce finitely many extra factors of  $x_j$  and derivatives of  $\chi(y_1) \cdots \chi(y_n)$ , and the  $x$ -derivatives produce extra factors of  $\chi(y_1) \cdots \chi(y_n)$ . All  $y$ -dependence can be uniformly estimated because of the compact support of (the derivatives of)  $\chi$ . So we arrive at a finite sum of the form

$$|\partial_x^\mu \partial_y^\nu \tau_x^n(y)_j| \leq \sum_{\nu' \leq \nu, \mu' \leq \mu} c_{\nu' \mu'} |x_j|^{s(\nu', \mu')} |(\partial_x^{\mu'} \partial_y^{\nu'} \tau^1)_{x_j \chi(y_1) \cdots \chi(y_n)}(y_j)|,$$

with  $s(\nu', \mu') \leq |\nu|$ . For  $y_j \in I$ , the derivatives of  $\tau^1$  can now be estimated with (5.11), and  $(1 + x_j^2) \leq (1 + \|x\|^2)$ . For  $y_j \in I_\varepsilon \setminus I$ , we can use the invariance  $\tau_t(y_j) = y_j$ ,  $t \in \mathbb{R}$ , and  $|y_j| \leq 1 + \varepsilon$ , to estimate the derivatives of  $\tau^1$ . This shows that if  $\tau^1$  was of order  $\mathbf{b}$ , then  $\tau^n$  is of order at most  $b_l + l < \infty$ .  $\square$

After these constructions, it is now easy to show the existence of smooth polynomially bounded  $\mathbb{R}^n$ -actions supported in arbitrarily small regions.

**Theorem 5.8** *Let  $K \subset \mathbb{R}^n$  be open. Then there exist non-trivial smooth polynomially bounded  $\mathbb{R}^n$ -actions on  $\mathbb{R}^n$ , with support in  $K$ .*

Proof. In Lemma 5.7, we have constructed a non-trivial smooth polynomially bounded  $\mathbb{R}^n$ -action  $\tau^n$  with support in a cube  $[-r, r]^{\times n}$  centered at the origin. Clearly, the polynomial estimates are at most rescaled by affine transformations of  $\mathbb{R}^n$  which allows to squeeze and move the support into any given compact subset.  $\square$

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